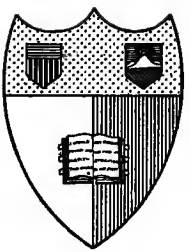
COMMERCIAL ALGEBRA

BOOK I



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WENTWORTH-SMITH MATHEMATICAL SERIES

COMMERCIAL ALGEBRA

BOOK I

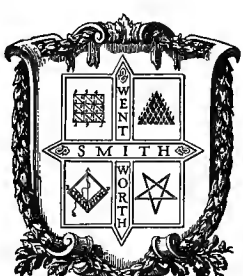
BY

GEORGE WENTWORTH

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AND

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PREFACE

This book is intended for use in the first year of a commercial high school. It lays the foundation for that work in mathematics without which it is impossible for a student to attain the highest success in the technique of commerce. A man may have a genius for organization, he may employ others to do his mathematical work for him, and he may advance through a combination of circumstances, but if he is to master the technique of commercial life he must be able to think in algebra. There is not to-day a single line of commercial work in which the formula is not, or may not be, profitably used; and without a knowledge of the formula, of its value, and of its manipulation by the aid of the equation, a man cannot be a real master of any great commercial field.

This knowledge of the formula is, however, no more vital than a knowledge of other parts of algebra, as, for example, the graph in its varied forms. Here is one of the several tools of mathematics which every business man needs and actually uses in his everyday business life. Algebra, then, may be said to be vital to successful training for commercial life.

It must not be thought, however, that the necessary algebra can be taught in a few lessons, or that only the actual utilities will suffice. A student need not attempt to master all the formalism which has found place in our traditional courses, but it is essential that he should acquire that power of manipulating simple algebraic expressions which comes only by actually doing the work. For this reason, while the authors have eliminated a large amount of non-essential work in factoring, fractions, and complicated equations, they have retained a sufficient amount of abstract work to assure the student of the drill which is necessary to his success.

Students who may be preparing for college will find that the work which is given in this book is a sufficient foundation for a subsequent half year of review, drill, and study of special topics like the quadratic, that will assure them of ability easily to enter any such institution.

The second book in this series is devoted largely to the higher technical topics of commercial algebra. The pursuit of such a work is essential to a complete mastery of the modern treatment of subjects like investments, depreciation funds, insurance, and exchange. Such subjects cannot be treated properly in a one-year course, and hence cannot be successfully presented in a book like this. Book II therefore includes the topics of powers and roots, logarithms, the slide rule, series, compound interest, equation of payments, annuities, amortization, depreciation, bond valuation, life insurance, and alignment charts; these being the ones that the student is most apt to need in general commercial work.

It is the hope of the authors that teachers who seek to raise our commercial high schools to the level of the best to be found in other countries will find in this book the material they seek for an elementary course in the important field of algebra.

CONTENTS

CHAPTER		PAGE
I.	Letters used in Formulas	1
II.	The Equation	29
III.	Elementary Graphs	43
IV.	Negative Numbers	57
V.	Addition and Subtraction	71
VI.	Multiplication	93
VII.	Division	119
VIII.	Simple Equations	131
IX.	Fractions	149
X.	Fractional Equations	163
XI.	Equations applied to Commerce	189
XII.	Simultaneous Simple Equations	221
XIII.	Graphs	245
INDEX		265

COMMERCIAL ALGEBRA

BOOK I

CHAPTER I

LETTERS USED IN FORMULAS

1. Use of Abbreviations. If we add 3 feet, 5 feet, and 7 feet, the result is 15 feet. Practically we use the abbreviation ft. for feet, and write 3 ft. + 5 ft. + 7 ft. = 15 ft. A carpenter will write this in a still simpler way, thus: $3' + 5' + 7' = 15'$. In the same way we may write 3 apples + 5 apples + 7 apples = 15 apples, or we may use the letter a to mean apples and write $3a + 5a + 7a = 15a$.

3 feet	3 ft.	$3'$	3 apples	$3a$
5 feet	5 ft.	$5'$	5 apples	$5a$
7 feet	7 ft.	$7'$	7 apples	$7a$
15 feet	15 ft.	$15'$	15 apples	$15a$

2. Symbols used for Numbers. We also commonly use symbols for numbers, as in the following examples:

3 doz.	3 gr.	3 M	3 C	3 R
5 doz.	5 gr.	5 M	5 C	5 R
7 doz.	7 gr.	7 M	7 C	7 R
15 doz.	15 gr.	15 M	15 C	15 R

Here we use doz. for dozen, or 12; gr. for gross, or 144; M for 1000, a custom of bond dealers and lumber dealers; C for 100, a custom in putting up envelopes; and R for 480 or 500, that is, a ream, a custom in the paper trade.

3. Letters used for Numbers. We have seen that symbols like doz., gr., M, C, and R are commonly used for particular numbers. But just as $3\,M + 5\,M + 7\,M = 15\,M$, where M stands for 1000, so $3x + 5x + 7x = 15x$, whatever number x represents. If $x = 10$, it is still true that $3x + 5x + 7x = 15x$, for 3 tens + 5 tens + 7 tens = 15 tens; and similarly for any other number that x may represent.

In business it is convenient to represent numbers by letters in marking goods, but more important uses will be found as we proceed with the study of algebra.

Exercise 1. Letters used for Numbers

All work oral

Add the following:

1.	2.	3.	4.	5.	6.
$2a$	$5x$	$7m$	$2y$	$8b$	$25c$
$3a$	$4x$	$3m$	$9y$	$6b$	$5c$

Subtract the following:

7.	8.	9.	10.	11.	12.
$9m$	$8d$	$7s$	$9x$	$8y$	$36c$
$4m$	$2d$	$3s$	$2x$	$3y$	$11c$

13. If $x = 10$, what is the value of $3x + 7x$?

We see that $3x + 7x = 10x$, or 3 tens + 7 tens = 10 tens = ( ).

14. If $n = 9$, what is the value of $8n - 2n$?

15. If $s = 7$, what is the value of $16s - 4s$?

16. If $e = 11$, what is the value of $16e - 8e$?

17. From 17 doz. subtract 9 doz.; from 17 ft. subtract 9 ft.; from 17 M subtract 9 M; from $17x$ subtract $9x$.

4. Simplifying. If we have the expression $15x + 6x - 3x$, we may first add $15x$ and $6x$, the sum being $21x$, and then subtract $3x$, the result being $18x$. We say that the expression has now been *simplified.*

Teachers should not ask for formal definitions at this time. It is merely necessary to use such a term as "simplify" intelligently.

5. Evaluating. If we have the expression $3m$ and know that the value of m is 7, we see that the value of $3m$ is 3×7, or 21. We have now *evaluated* $3m$ for $m = 7$. To evaluate $3m$ for $m = 4$ we have $3 \times 4 = 12$.

Exercise 2. Simplifying and Evaluating

Examples 1 to 10, oral

Simplify the following expressions:

1. $3a + 2a + 7a$.
2. $6b + 7b - 2b$.
3. $8m + 9m - 3m$.
4. $5x + 8x + 9x$.
5. $9y + 9y - 7y$.
6. $8p + 7p + 9p$.
7. $5q + 9q - 8q$.
8. $7r + 7r - 6r$.
9. $9n + 9n - 2n$.
10. $6c + 7c + 8c$.
11. $15x + 6x - 7x$.
12. $18y + 9y - 6y$.
13. $27m + 13m + 10m$.
14. $62n + 12n - 15n$.
15. $75a + 5a - 2a$.
16. $80a - 10a - 10a$.
17. $32b + 32b + 32b$.
18. $68c + 32c - 90c$.
19. $56d + 44d + 11d$.
20. $128m + 275m$.
21. If $x = 15$, find the value of $25x + 20x + 5x$.
22. Evaluate the expression $27x + 3x - 15x$ for $x = 12$.
23. Evaluate the expression $81 \div x$ for $x = 9$.
24. Evaluate the expression $x \div 5$ for $x = 75$.

6. Uses of Other Symbols. Many other symbols besides letters are used in business records, and a commercial student must become familiar with the advantage of such use. For example, there is shown below a daily record card sent by a salesman to the jobbing house which employs him.

Date	Firms Visited	Coal Ranges		Gas Ranges	Oil Stoves	Refrig-erators		Sundries
10/5	Jones & Co.	N		O	K	O	3/5	
	Coe & Conover . . .	+						
	J. & J. Straus . . .	O	2	O	L	K		O
	L. B. Smith	L			O	t		
	A. L. Bowles . .	O	1	N		O	4/6	
	T. Trask & Bro.	+			K			
	Johns & Johns . . .	=			K			

This jobbing house instructs its salesmen to use the following key: =, called; +, out; *t*, intend; O, order; N, not in the line; L, lost order; K, left catalogue; 1, leg; 2, base; 3, oak; 4, chestnut; 5, porcelain; 6, zinc.

This card shows at once that the salesman made seven calls on Oct. 5; that Jones & Co. are not dealing in coal ranges, that they gave an order for gas ranges, that a catalogue of oil stoves was left, and that they gave an order for oak refrigerators lined with porcelain; that the buyer of coal ranges for Coe & Conover was out; and so on.

Thus, by the aid of a few symbols a large amount of information is given on a small card, the salesman using a kind of shorthand.

Exercise 3. Commercial Symbols

1. The salesman referred to on page 4 sent in five other cards for the rest of the week beginning Oct. 5. On Oct. 6 he called on White & Weld, secured an order for coal ranges with legs, and found that they did not handle gas ranges, oil stoves, or refrigerators. Wolcott & Co. intended to buy some coal ranges and oil stoves, but refrigerators were not in their line. Leach & Co. gave an order for coal ranges with base, intended to order some gas ranges, did not handle oil stoves, and ordered some chestnut refrigerators with zinc lining. The American Hardware Co. ordered some coal ranges with base and some gas ranges, and took a catalogue on oil stoves, but they did not handle refrigerators. The salesman called on C. R. Weeks & Son, left a catalogue of gas ranges, but made no sales. He then called on the J. M. Caldwell Co. and secured an order for oil stoves. Make out a card as on page 4, giving all this information in condensed form.

2. Make out cards for the other four days of the week, as in Ex. 1, giving imaginary names, sales, and information.

3. From the card given on page 4 and from those prepared in Exs. 1 and 2, add the +'s, t's, O's, N's, L's, and K's, give the results, and write out a statement telling what each means to the jobbing house.

4. On the following week the cards showed, under coal ranges, O, L, =, +, K, O, O, +, K, O, O, t, O, =, O, O, t, O, =, K, O, L, =, +, K. Insert the O's, L's, +'s, and so on, give the results, and tell what the record means.

5. In the same week as in Ex. 4 the cards showed, under refrigerators, O, K, O, N, N, O, N, N, K, O, O, O, N, =, O, O, O, O, t, K, t, N. Add the like symbols and tell what the record means.

Exercise 4. Review

Examples 1 to 16, oral

Evaluate the following for C = 100:

1. 2 C. **2.** 7 C. **3.** $\frac{1}{2}$ C. **4.** 25 C. **5.** $1\frac{1}{2}$ C.

Evaluate the following for M = 1000:

6. 4 M. **7.** 6 M. **8.** $\frac{1}{4}$ M. **9.** 10 M. **10.** $2\frac{1}{2}$ M.

Simplify the following:

11. $2x + 7x$. **13.** $3m + 9m$. **15.** $17a + 3a$.

12. $9c - 4c$. **14.** $12a - 9a$. **16.** $27n - 10n$.

17. Evaluate the expression $7x + 9x$ for $x = 12$.

18. Simplify the expression $17x + 29x - 3x$.

19. Simplify $25a + 17a - 12a$ and evaluate for $a = 5$.

20. Evaluate 15 R for R = 480.

21. A dealer buys 150 M envelopes. Taking M to mean 1000, as usual, how many envelopes did he buy, and how much did they cost at \$2.75 per M?

22. A dealer sells four lots of bonds, as follows: 25 M, 15 M, 30 M, and 12 M. If his commission is \$1.25 per M, find his total commission on the four lots.

23. In Ex. 22 find the dealer's commission on each lot separately, add the results, and thus check the answer.

24. A contractor buys 12 M bricks at \$8.50 per M, 15 M at \$8.67 per M, and 8 M at \$9.20 per M. How many bricks did he buy? How much did he pay in all?

25. A hardware dealer bought 2 gr. (2 gross) of hinges at \$12.60, 3 gr. at \$8.20, and 7 gr. at \$4.20. How many hinges did he buy? How much did he pay in all?

7. Nature of Arithmetic. In arithmetic we usually represent numbers by figures. If the page of a book is seven inches long and four inches wide, we represent these dimensions by 7 in. and 4 in. respectively. In finding the area, we say that

$$\text{area} = 4 \times 7 \text{ sq. in.} = 28 \text{ sq. in.}$$

8. Nature of Algebra. We might state the process of finding the area of the page as follows:

$$\text{area} = \text{width} \times \text{length},$$

meaning that the number of square inches of area is equal to the product of the number of inches of width and the number of inches of length. We do this in algebra, often using initial letters, thus:

$$a = w \times l.$$

Usually in algebra we indicate the multiplication of one letter by another by writing the letters without the sign of multiplication; thus wl means $w \times l$.

Arithmetically stated: *The number of units of area of a rectangle is equal to the product of the number of units of width and the number of units of length.*

Algebraically stated: $a = wl$.

One use of algebra, therefore, is to make short statements of the rules of arithmetic. It is thus seen that algebra is a kind of shorthand, enabling us to give a large amount of information in a small space.

We have now learned a new fact about the use of letters in algebra, namely, that *if two letters are written side by side, as in a word, the product of their values is to be taken.* That is:

If $a = 4$ and $b = 3$, then $ab = 4 \times 3 = 12$;

if $a = 7$ and $b = 9$, then $ab = 7 \times 9 = 63$;

and if $a = \frac{3}{4}$ and $b = 36$, then $ab = \frac{3}{4} \times 36 = 27$.

9. Arithmetic and Algebraic Statements. We may compare arithmetic and algebraic statements as follows:

Arithmetic	Algebra
If 1 book costs \$2, 3 books cost $3 \times \$2$.	If 1 book costs d dollars, n books cost nd dollars.
The cost of any number of books is equal to the cost of one book multiplied by the given number of books.	If c represents the total cost, then $c = nd$.
If a train travels 35 mi. an hour, in 4 hr. it travels 4×35 mi., or 140 mi.	If a train travels m miles an hour, in h hours it travels hm miles.
To find the distance traveled by a train in any number of hours, multiply the number of miles per hour by the number of hours.	If d represents the total distance, then $d = hm$.

10. Formula. A rule stated algebraically, in letters, is called a *formula.*

For example, $c = nd$ and $d = hm$, given above, are formulas.

In arithmetic we learn that the volume of a box is equal to the product of the length, width, and height. In algebra we represent this by the formula

$$v = lwh.$$

In this case, if $l = 3$, $w = 2$, and $h = 1$, we have

$$v = 3 \times 2 \times 1 = 6.$$

In writing algebraic formulas it is not customary to express denominations like feet and inches.

Before considering commercial formulas we shall take up a number of formulas relating to measurements with which all students are familiar. In this way we shall introduce the commercial work more easily.

Exercise 5. Formulas

Examples 1 to 9, oral

1. If a rectangle is 8 in. long and 4 in. wide, how many square inches of area does it contain?

2. If a rectangle has a base of 12 in. and a height of 4 in., what is its area?

3. If a rectangle has a base of b inches and a height of h inches, what is its area?

4. If a rectangle is l inches long and w inches wide, how many square inches of area does it contain?

5. If 1 yd. of velvet costs \$2, how much will 9 yd. cost? If 1 yd. costs d dollars, how much will y yards cost?

6. At the rate of 3 mi. an hour, how far will a man walk in 2 hr.? At the rate of m miles an hour, how far will he walk in h hours?

7. Read from the formula $a = mn$ the rule for finding the area of a rectangular field m rods wide and n rods long.

8. Read from the formula $v = pqr$ the rule for finding the volume of a rectangular box p inches long, q inches wide, and r inches deep.

9. Read from the formula $n = c \div e$ the rule for finding the number of articles purchased when c, the cost of all, and e, the cost of each, are given.

10. Write a formula for d, the average cost of each of n things, when they cost c dollars in all.

11. Write a formula for c, the cost of f feet of iron pipe, at n cents a foot. Write the formula so as to express the result in cents; in dollars.

12. If n is any integer, is $2n$ an even or an odd number? Why? What is the value of $2n$ when $n = 197$? 276? 997?

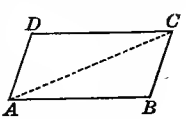
11. Formula for the Area of a Parallelogram. It is shown in arithmetic that the area of a parallelogram, like the area of a rectangle, is equal to the product of the base and height. We may therefore express this statement in algebraic form thus: $a = bh$.



For the triangle T may be cut off and placed at X, so as to make a rectangle of area bh.

If $b = 5$ and $h = 3$, then $a = bh = 5 \times 3 = 15$. If b and h represent inches, then a represents square inches. When we speak of the product of two lines we mean the product of their numerical values.

12. Formula for the Area of a Triangle. It is also shown in arithmetic that the area of a triangle is equal to half the product of the base and height. This statement may be expressed in algebraic form thus: $a = \frac{1}{2}\,bh$.



This triangle may be cut as here shown so that it is seen to be half of the parallelogram of base b and height h.

If $b = 7$ and $h = 10\frac{1}{2}$, then $a = \frac{1}{2}\,bh = \frac{1}{2}$ of $7 \times 10\frac{1}{2} = 36\frac{3}{4}$.

Exercise 6. Parallelograms and Triangles

Examples 1 to 6, oral

1. If $b = 4$ and $h = 3$, what is the value of bh? of $\frac{1}{2}\,bh$?
2. If $b = 60$ and $h = 5$, what is the value of bh? of $\frac{1}{2}\,bh$?

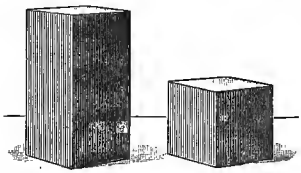
Given $a = bh$, find the value of a when:

3. $b = 60$, $h = 11$.
4. $b = 45$, $h = 10$.
5. $b = 20$, $h = 12$.
6. $b = 22$, $h = 10$.

Given $a = \frac{1}{2}\,bh$, find the value of a when:

7. $b = 48$, $h = 25$.
8. $b = 36$, $h = 19\frac{1}{2}$.
9. $b = 24.8$, $h = 4.75$.
10. $b = 63.2$, $h = 19.65$.

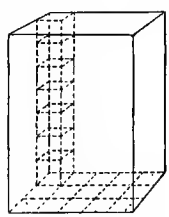
13. Rectangular Solid. A solid bounded by six rectangles is called a *rectangular solid.* A rectangular solid all of whose faces are squares is called a *cube.*



The six rectangles are called the *faces* of the solid, and their sides form the *edges* of the solid.

A rectangular solid has length, breadth, and thickness, and these are called the *dimensions* of the solid.

14. Formula for the Volume of a Rectangular Solid. If this figure represents a rectangular solid 5 in. long, 3 in. wide, and 7 in. high, it is evident that in the column of cubes shown there are 7 cu. in. It is also evident that on the base we can place 3×5 such columns. Therefore the volume is $3 \times 5 \times 7$ cu. in., or 105 cu. in. Hence,



The volume of a rectangular solid is equal to the product of the three dimensions.

We may express this rule in the compact algebraic formula

$$v = lwh.$$

Exercise 7. Rectangular Solids

Given $v = lwh$, find the value of v when:

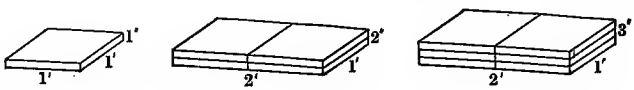
1. $l = 8,\ w = 5,\ h = 3.$
2. $l = 6,\ w = 5.2,\ h = 2.3.$
3. $l = 17,\ w = 8.4,\ h = 7.9.$
4. $l = 34,\ w = 17\frac{1}{2},\ h = 12\frac{3}{4}.$

5. A rectangular solid is 33 ft. long, 5 yd. wide, and 16 in. high. Find the volume in cubic feet; in cubic yards.

6. A cellar is 24 ft. by 36 ft. by $7\frac{1}{2}$ ft. Find the volume in cubic feet; in cubic yards.

7. A room is 22 ft. 8 in. long, 16 ft. 4 in. wide, and 9 ft. 3 in. high. Find the volume in cubic feet.

15. Board Measure. A *board foot* is the measure of a piece of lumber 1 sq. ft. on one face and 1 in. or less in thickness.



1 BOARD FOOT 4 BOARD FEET 6 BOARD FEET

For example, a board 2′ long, 1′ wide, and 1″ (or less) thick contains 2 board feet (usually written as 2 bd. ft.).

A joist $12' \times 6'' \times 2\frac{1}{2}''$ contains $2\frac{1}{2} \times 12 \times \frac{1}{2}$ bd. ft., or 15 bd. ft.

A joist $12' \times 6'' \times 2\frac{1}{2}''$ means a joist 12 ft. by 6 in. by $2\frac{1}{2}$ in.

In such cases usually use cancellation; thus, since $2\frac{1}{2} = \frac{5}{2}$, we have

$$2\tfrac{1}{2} \times 12 \times \frac{1}{2} = \frac{5 \times \overset{3}{\cancel{12}}}{\cancel{2} \times \cancel{2}} = 15.$$

If we let F represent the number of board feet, l the length in feet, w the width in inches, and t the thickness in inches, we have $F = \frac{1}{12}\, lwt$.

Exercise 8. Formula for Board Measure

Copy the following table, supply the missing numbers marked x, and find the total value at $48 per M:

Lot Number	l Feet	w Inches	t Inches	F in Each Piece	Number of Pieces	F Total
1	16	6	3	24	50	1200
2	12	9	4	x	40	x
3	16	18	6	x	20	x
4	12	9	1	x	500	x
5	16	3	1	x	300	x
Total number of board feet						x

16. Terms used in Algebra and Arithmetic. As already seen, many of the terms of mathematics are used in algebra exactly as in arithmetic. Since they are well known, such terms usually do not require further definition, although a few are formally defined at this time for future reference.

It should always be remembered that the letters of algebra represent numbers. This is the reason why the terms used in arithmetic may properly be employed in connection with algebraic expressions.

17. Factor. Any one of two or more numbers is called a *factor* of the product of those numbers.

Thus just as 2 and 3 are factors of 6, so a and b are factors of ab.

A factor that has itself no factors is called a *prime factor*.

Thus 4 and 3 are factors of 12, but the prime factors are 2, 2, and 3. Neither the number itself nor 1 is considered as a factor of a number.

18. Literal and Numerical Factors. A factor that contains a letter is called a *literal* factor; a factor that is expressed by a numeral is called a *numerical* factor.

Factors are generally considered to be integers; but in an expression like $2ab$, for example, a and b are factors although their numerical values may be fractional.

19. Coefficient. If an expression is the product of two factors, either factor is called the *coefficient* of the other.

Thus in the expression ab, a is the coefficient of b, and b is the coefficient of a. The factor that is considered the coefficient is usually written first, however. Thus in $2ab$, 2 is the coefficient of ab, and $2a$ is the coefficient of b. The coefficient 1 is omitted, x being the same as $1x$.

20. Power. The product of several equal factors is called a *power* of any one of the equal factors.

Thus $2 \times 2 = 2^2$, or 4; and 2^2, or 4, is called the *square* or *second power* of 2. Also $2 \times 2 \times 2 = 2^3$, or 8; and 2^3, or 8, is called the *cube* or *third power* of 2. Similarly, aa is the *square* or *second power* of a, and is written a^2; and aaa is the *cube* or *third power* of a, and is written a^3.

21. Exponent. The number or letter placed to the right and slightly above another to indicate how many times that number or letter is taken as a factor is called an *exponent.*

Thus in $2r^3$, 2 is the coefficient of r^3, and 3 is the exponent of r.

The two should be carefully distinguished. *The coefficient shows the number of equal addends; the exponent shows the number of equal factors.*

22. Root. Any one of the equal factors whose product is a given expression is called a *root* of the expression. One of the two equal factors whose product is a given expression is called the *square root* of the expression; one of the three equal factors is called the *cube root*; one of the four equal factors is called the *fourth root*; and so on.

Exercise 9. Terms Used

All work oral

State the factors of the following numbers:

1. 15. **2.** 77. **3.** 35. **4.** 14. **5.** 33.

State any two factors of the following numbers:

6. 70. **7.** 80. **8.** 100. **9.** 120. **10.** 1250.

State the two equal factors of each of the following numbers:

11. 16. **12.** 36. **13.** 81. **14.** 121. **15.** 144.

State the prime factors of each of the following numbers:

16. 32. **17.** 20. **18.** 48. **19.** 50. **20.** 75.

State the coefficient of x in each of the following cases:

21. $17x$. **22.** x. **23.** $\frac{1}{2}x$. **24.** $1.5x$. **25.** $2ax$.

State the exponent of a in each of the following cases:

26. $2a^3$. **27.** $3a^2$. **28.** $\frac{1}{2}a^4$. **29.** a^{10}. **30.** $37\frac{1}{2}a^9$.

23. Symbols. We may now summarize the symbols of operation which are most commonly used to indicate the operations used in commercial algebra:

	Arithmetic	Algebra
Addition	$4 + 3$	$a + b$
Subtraction	$4 - 3$	$a - b$
Multiplication	4×3	$a \times b$, $a \cdot b$, or ab
Division	$4 \div 3$	$a \div b$, a/b, $a : b$, or $\frac{a}{b}$
Second power	5^2	a^2
Third power	5^3	a^3

A letter without an exponent is considered as having the exponent **1**. That is, a^1 and a have the same meaning.

Square root	$\sqrt{4}$ means the square root of 4.	$\sqrt{a}$ means the square root of a.
Cube root	$\sqrt[3]{27}$ means the cube root of 27.	$\sqrt[3]{b}$ means the cube root of b.

Since the square root of a number is one of the two equal factors of the number, it follows that the square root of 9 is 3; similarly, the third root (always called the cube root) of 8 is 2, and since $2^4 = 16$, it follows that 2 is the fourth root of 16.

If a is not a perfect square, as 3, 5, or 7, then $\sqrt{a}$ can be expressed only approximately as a decimal. This is always done in practice, and similarly for other roots.

Of the various symbols of division, the form a/b is coming to be very common in print and in typewritten work.

It is customary to use the symbol $\therefore$ for "therefore," $=$ for "is equal to," $<$ for "is less than," and $>$ for "is greater than" whenever it is necessary to save space.

Such commercial symbols as a/c, f.o.b., $N/20$, C.O.D., and the like either are already familiar to the student or will be explained as they are needed in the progress of the work.

24. Monomial. An algebraic expression in which the parts are not separated by the signs + or − is called a *monomial.*

Thus a, $3ab$, a^2, and $\sqrt{a}$ are monomials.

25. Polynomial. An algebraic expression consisting of two or more monomials separated by the signs + or − is called a *polynomial.*

Thus $a + b$, $3a - 4b$, $2n^2 + 1$, and $a^2 - \frac{1}{2}b^3 + c$ are polynomials.

26. Terms of a Polynomial. The monomials that make up a polynomial are called the *terms* of the polynomial.

A polynomial is called a *binomial* if it has only two terms, and a *trinomial* if it has only three terms.

Thus $a + 3b$ is a binomial and its terms are a and $3b$; $a + 3b + c$ is a trinomial and its terms are a, $3b$, and c.

27. Symbols of Aggregation. Symbols used to indicate that certain terms are to be treated as one number or one quantity are called *symbols of aggregation.*

The most common of these are the parentheses, brackets, and bar. Others will be given when needed. Thus $2 \times (5 + 7)$, $2 \times [5 + 7]$, or $2 \times \overline{5 + 7}$ means that we are to add 5 and 7 before we multiply by 2; each is read "twice the sum of 5 and 7," and each is equal to 24.

28. Order of Operations. Unless symbols of aggregation direct otherwise, the operations indicated in an algebraic expression are performed in the following order:

1. *Powers and roots.*

2. *Multiplications and divisions in the order in which they occur.*

3. *Additions and subtractions in the order in which they occur or in any other order.*

In the case of $a^2b + c \div d$ we first square a and take the product of a^2 and b; we then divide c by d; and finally we add the results. Thus

$$2^2 \times 3 + 10 \div 5 = 4 \times 3 + 10 \div 5 = 12 + 2 = 14.$$

Similarly, $\sqrt{3 + 6} + 15 \div 5 - 2 + 7 \times 6 = 3 + 3 - 2 + 42 = 46.$

Exercise 10. Algebraic Expressions

Examples 1 to 13, oral

1. Read $4+7$; $a+b$; $3a+b$; $3a+5b$.

2. Read $9-5$; $a-b$; $7a-b$; $7a-4b$.

3. Read 2×7; $a\times b$; ab; $3ab$; $a(b+c)$; $x(a+b+c)$.

4. Read $9\div 3$; $a\div 3$; $a\div b$; $3a\div b$; $9a/b$; $9a/7$; $10ab/c$.

5. Read $2n$; $2n+1$; $2n-1$. What is the value of each when $n=3$? when $n=5$? when $n=10$?

6. Read $\frac{1}{2}n$; $\frac{1}{3}n$; $\frac{2}{3}n$; $\frac{3}{4}n$. What is the value of each when $n=12$? when $n=24$? when $n=10$?

7. How many feet in a yard? in 5 yd.? in n yards?

8. How many inches in a foot? in 2 ft.? in n feet?

9. How many ounces in a pound? in 10 lb.? in n pounds?

10. Read $a+a=2a$; $a<2a$; $2a>a$; 1 gal. > 1 qt.

11. If 1 lb. of tea costs $\$\frac{1}{2}$, how much will n pounds cost?

12. Read 2 ft.; 2 yd.; $2k$; $2x$; 2 mi.; $2m$; 2 A.; $2a$.

13. Read 3 ft. 2 in.; $3x+2y$; $3a+2b$; $\$3+2\phi$.

14. Write the sum of 2 and y; of x and y; of $2x$ and y.

15. Indicate the subtraction of y from 7; of y from x.

16. Write these products: 2 times x; 2 times y; x times y.

17. Write the quotient of x divided by 2, both as a fraction and with some other sign of division.

18. In the same two ways as in Ex. 17 write the quotient of $2x$ divided by 3; of $2x$ divided by y; of $2x$ divided by $3y$.

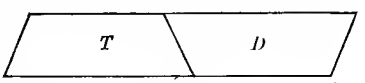
19. Write the sum of $5x^2$, $7x$, and 2. What is its value when $x=10$? What is its value when $x=20$?

20. Write the sum of $7x^3$, $2x^2$, $3x$, and 6. What is its value when $x=10$? What is its value when $x=20$?

29. Trapezoid. A plane figure of four sides which has two of its sides parallel is called a *trapezoid.*

30. Formula for the Trapezoid. It is usually shown in arithmetic that the area of a trapezoid is equal to half the product of the height by the sum of the two parallel sides.

For a trapezoid D, equal to the given trapezoid T, may be turned over and put down by the side of T, as here shown. The whole figure, or twice the trapezoid, is then equal to a parallelogram whose base is the sum of the parallel sides of the trapezoid. The trapezoid is therefore half this parallelogram.



We indicate this in algebraic form as follows:

$$a = \tfrac{1}{2} h(b + b'),$$

where a is the area, h the height, and b and b' ("b prime") the two parallel sides, usually called the *bases.*

For example, if $h = 4$, $b = 6$, and $b' = 5$, we have

$$a = \tfrac{1}{2} h(b + b') = \tfrac{1}{2} \times 4 \times (6 + 5) = \tfrac{1}{2} \text{ of } 4 \times 11 = 22.$$

Exercise 11. Trapezoids

Examples 1 to 5, oral

1. Find the value of $b + b'$ when $b = 7$, $b' = 9$.
2. Find the value of $h(b + b')$ when $h = 3$, $b = 4$, $b' = 5$.

Given $a = \tfrac{1}{2} h(b + b')$, *find the value of* a *when:*

3. $h = 3$, $b = 6$, $b' = 4$.
4. $h = 7$, $b = 9$, $b' = 5$.
5. $h = 6$, $b = 10$, $b' = 8$.
6. $h = 7.2$, $b = 24$, $b' = 9.5$.
7. $h = 19$, $b = 34$, $b' = 16$.
8. $h = 8.6$, $b = 3.8$, $b' = 9.8$.

9. A playground is in the form of a trapezoid, with bases x and y and with height z. What is the area? How many square rods are there in the playground if $x = 30$ rd., $y = 26$ rd., and $z = 24$ rd.? if $x = 34$ rd., $y = 28$ rd., and $z = 30$ rd.?

Exercise 12. Commercial Formulas

Examples 1 to 4, oral

1. At 6% a year, what is the interest on \$200 for 1 yr.?

2. If r is the rate for 1 yr., what is the interest on p dollars for 1 yr.? for 2 yr.? for $2\frac{1}{2}$ yr.?

3. At 6% a year, what is the interest on \$200 for $2\frac{1}{2}$ yr.?

4. If r is the rate for 1 yr., what is the interest on p dollars for t years?

5. If the list price of some goods is \$575 and a discount of 10% is allowed, what is the discount?

6. If the list price of some goods is l dollars and the rate of discount is r, what is the discount?

7. If some stock is selling at 7% below par, what is the cost of \$1500 worth, par value, brokerage not considered?

8. If some stock is selling at r% below par, what is the cost of p dollars' worth, par value, brokerage not considered?

9. What is the amount of principal and interest of \$1700 for 2 yr. at 5%?

10. What is the amount of principal and interest of p dollars for t years, the rate of interest being r?

11. A trade price list gives the cost in dollars per foot of sewer pipe of diameter d inches, as follows: $c = 0.004d^2 + 0.14$. Find the cost of 300 ft. of 16-inch pipe.

12. From the formula of Ex. 11 find the cost of half a mile of 18-inch pipe.

13. As in Ex. 11 find the cost of 30 rd. of $8\frac{1}{2}$-inch pipe.

14. If the amount of d dollars on interest for n years at the rate r is $d(1 + nr)$, find the amount of \$750 for 3 yr. 6 mo. at 5% and the amount of \$625 for 2 yr. 9 mo. at 6%.

31. Table of Squares and Cubes. We frequently need the squares and cubes of numbers in our evaluation of expressions, and hence the following table should be committed to memory:

No.	Square	Cube	No.	Square	Cube	No.	Square	Cube
1	1	1	6	36	216	11	121	1331
2	4	8	7	49	343	12	144	1728
3	9	27	8	64	512	13	169	
4	16	64	9	81	729	14	196	
5	25	125	10	100	1000	15	225	

32. Raising Numbers to Powers. In commercial work we frequently need to raise a number like 1.06 to some power, say to find 1.06^5.

Since $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 2^2 \times 2^3 = 4 \times 8 = 32$, we see that we can find the value of 1.06^5 as follows:

$$
\begin{array}{rl}
1.06 & 1.191016 = 1.06^3 \\
1.06 & 1.1236 = 1.06^2 \\
\hline
6\ 36 & 7146096 \\
106 & 3573048 \\
\hline
1.1236 = 1.06^2 & 2382032 \\
1.06 & 1191016 \\
\hline
67416 & 1191016 \\
11236 & \hline \\
\hline
1.191016 = 1.06^3 & 1.3382255776 = 1.06^5
\end{array}
$$

In elementary business matters no such number of decimal places would be needed, but in large commercial and banking transactions involving millions of dollars they are frequently found.

The student will also find it convenient to accustom himself to multiplying from left to right instead of following the elementary plan.

Exercise 13. Evaluation

Examples 1 to 15, oral

State the value of each of the following:

1. 11^2.	**3.** 12^2.	**5.** 13^2.	**7.** 14^2.	**9.** 15^2.
2. 11^3.	**4.** 12^3.	**6.** 10^3.	**8.** 10^2.	**10.** 100^3.

Evaluate the following expressions for $a = 4$, $b = 2$, $c = 3$:

11. a^2.	**12.** c^2.	**13.** a^2b.	**14.** $2\,ab$.	**15.** c^3.

Find the values of the following:

16. 2^4.	**17.** 2^7.	**18.** 7^4.	**19.** 12^4.	**20.** 120^2.

Evaluate the following expressions for $x = 2$, $y = 0$, $z = 5$:

21. $2y$.	**22.** y^2.	**23.** xz.	**24.** xz^2.	**25.** x^3z^2.

26. If $x = 3$ and $y = 2$, find the value of $5x^3y^4$.

Evaluate the following expressions for $x = 2$, $y = 3$, $z = 4$:

27. $3x^2y$.	**28.** $7xyz$.	**29.** $15x^2yz^2$.	**30.** $6x^3y^3z$.

Find the values of the following:

31. $2^3 \times 2^5$.	**32.** 2^9.	**33.** 1.04^5.	**34.** 1.045^4.

35. To what power of 2 is $2^5 \times 2^6$ equal? Find the value.

36. To what power of x is x^3x^7 equal? Why is this?

37. By carrying each multiplication to three decimal places only, find the value of 1.03^8 to the nearest thousandth.

Proceed as in § 32; $1.03^2 \times 1.03^2 = 1.03^4$ and $1.03^4 \times 1.03^4 = 1.03^8$.

38. What is the error to the nearest thousandth in retaining only three decimal places in each multiplication used in finding 1.05^4 instead of keeping all the figures?

33. Compound Interest. We are now prepared to understand another very important formula used in business.

In finding *compound interest* the simple interest is added to the principal and becomes part of it whenever it is due.

The sum of the principal and compound interest is called the *compound amount.*

Savings banks usually add the interest to the principal every 6 mo. Banks of deposit continually lend their interest as it is paid in and thus receive the benefit of compound interest.

For example, if \$1 is put at interest for 1 yr. at 6%, it amounts to \$1.06 at the end of the year.

Now if you started the second year with, say, \$1, the amount at the end of the second year would again be \$1.06, and if you started it with \$20 it would be $20 \times \$1.06$.

But we have seen that you really start the second year with \$1.06, and so its amount at the end of the second year is $1.06 \times \$1.06$, or $\$1.06^2$. Hence we see that

the amount of \$1 at the end of 1 yr. is \$1.06,
the amount of \$1 at the end of 2 yr. is $\$1.06^2$,
the amount of \$1 at the end of 3 yr. is $\$1.06^3$,
and the amount of \$1 at the end of n years is $\$1.06^n$;

or, if r is the rate per cent,

the amount of d dollars at the end of n years is $d(1+r)^n$.

Hence, if p is the principal, A the amount, and r the rate,

$$A = p(1+r)^n.$$

For example, to find the compound amount of \$900 for 5 yr. at 4%, we have

$$A = 900 \times 1.04^5 = 900 \times 1.21665 = 1094.99.$$

Hence the compound amount is \$1094.99.

Notice that, in this case, it is not necessary to find 1.04^5 to more than five decimal places to obtain the answer correct to the nearest cent.

Exercise 14. Compound Interest

Using the formula $A = p(1+r)^n$, find the amount of:

1. \$450, for 2 yr., at 3%.
2. \$920, for 3 yr., at 4%.
3. \$1200, for 3 yr., at 3%.
4. \$1500, for 4 yr., at 4%.

5. A certain savings bank compounds the interest on its depositors' accounts semiannually. If the rate is $3\frac{1}{2}\%$ a year, the rate for 6 mo. is $1\frac{3}{4}\%$. Therefore the compound amount for 2 yr. at $3\frac{1}{2}\%$ compounded semiannually is the same as the compound amount for 4 yr. at $1\frac{3}{4}\%$ compounded annually. If a man deposits \$500 on Jan. 1, how much is the compound amount 2 yr. hence?

We have \$500 multiplied by 1.0175^4.

6. Find the compound amount of \$850 for 3 yr. at 4%, the interest being compounded semiannually.

7. Since the compound interest is equal to the compound amount less the principal, we have $i = p(1+r)^n - p$. Find the value of i when $p = 400$, $r = 4\%$, and $n = 3$.

Find the compound interest on the following:

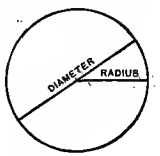
8. \$800, for 3 yr., at 5%.
9. \$1200, for 4 yr., at 4%.

10. Make a table of compound amounts of \$1 to the nearest mill, at the rates specified below, from 1 yr. to 6 yr. The first part of the table is as follows:

YEARS	2%	3%	$3\frac{1}{2}\%$	4%	5%	6%
1	1.020	1.030	1.035	1.040	1.050	1.060
2	1.040	1.061	1.071	1.082	1.103	1.124
3	1.061	1.093	1.109	1.125	1.158	
4	1.082	1.126	1.148			

34. Formula for the Circumference. We shall now consider a few further formulas of mensuration needed in commercial work. It is usually shown in arithmetic that the circumference of a circle is equal to nearly 3.1416 × the diameter. In mathematics the number 3.1416 −, or 3.14159 +, which is nearly $3\frac{1}{7}$, is represented by the Greek letter π (pronounced *pī*). We may therefore express this law as follows:

$$c = \pi d,$$



where c is the circumference and d the diameter.

In solving problems, take $3\frac{1}{7}$ for π unless otherwise directed.

Since the diameter is twice the radius, we may write $2r$ for d, and have $c = \pi \times 2r$, or $c = 2\pi r$.

Thus if $d = 7$, and we take $3\frac{1}{7}$, or $\frac{22}{7}$, as the value of π, we have

$$c = \pi d = 3\tfrac{1}{7} \times 7 = 22.$$

If $r = 5$, we have $c = 2\pi r = 2 \times 3\frac{1}{7} \times 5 = 10 \times 3\frac{1}{7} = 31\frac{3}{7}$.

Exercise 15. The Circumference

Given $c = \pi d = 2\pi r$, find the value of c when:

1. $d = 14$. 2. $d = 3.5$. 3. $r = 77$. 4. $r = 6.3$.

Taking $\pi = 3.1416$, find the value of c when:

5. $d = 10$. 6. $d = 50$. 7. $r = 30$. 8. $r = 2\frac{1}{2}$.

9. A boy measures the diameter of his bicycle wheel and finds it to be 28 in. Find the circumference.

10. A workman measures the diameter of a steel shaft and finds it to be $3\frac{7}{16}$ in. Find the circumference.

11. The basin of a fountain has a diameter of 30 ft. At 90¢ a foot, how much will it cost for an iron railing to inclose it?

35. Formula for the Area of a Circle. It is usually shown in arithmetic that

1. *The area of a circle is equal to half the product of the circumference and radius; that is,* $a = \frac{1}{2}cr$.

2. *The area of a circle is equal to* π *times the square of the radius; that is,* $a = \pi r^2$.

Thus if $r = 10$, we have $a = \pi r^2 = 3.1416 \times 10^2 = 314.16$, taking 3.1416 for π. If r represents the number of feet, the area is in square feet.

Exercise 16. Area of a Circle

Examples 1 to 3, oral

1. If $r = 1$ and $c = 6\frac{2}{7}$, what is the value of cr? of $\frac{1}{2}cr$?

2. If $r = 10$ and $c = 63$, what is the value of cr? of $\frac{1}{2}cr$?

3. If $r = 1$, what is the value of r^2? of πr^2?

Given $a = \frac{1}{2}cr$, *find* a *when:*

4. $r = 5$, $c = 31.416$.

5. $r = 2\frac{1}{2}$, $c = 15.708$.

Given $a = \pi r^2$, *find* a *when:*

6. $r = 7$.

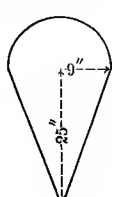
7. $r = 14$.

8. $r = 2.8$.

9. $r = 4.9$.

10. How many square feet in the area of a circle whose radius is 2 ft.? Take $\pi = 3.1416$.

11. How many square inches are there in the area of this kite? Take $\pi = 3\frac{1}{7}$; and also find the result when 3.1416 is taken for π.



12. Find the area covered by the basin of a circular fountain, the diameter being 30 ft.

13. The basin in Ex. 12 is surrounded by a walk 10 ft. wide. Find the cost of laying this walk at the rate of \$1.10 per square yard.

Exercise 17. Formulas of Measurement

Examples 1 to 6, oral

1. It is usually shown in arithmetic that the volume v of a cylinder is equal to the product of the base b and height h. Express this in a formula.

2. It is also shown that the volume is equal to the product of the height and π times the square of the radius of the base. Express this in a formula.



Express the following statements in formulas:

3. The volume v of a cube is equal to the third power of an edge e.

4. The entire surface s of a cube is equal to six times the second power of an edge e.

5. The lateral (or side) area of a cylinder is equal to the product of the circumference and height. Use l for lateral area, c for circumference, and h for height.

6. The lateral area of a cylinder is equal to 3.1416 times the diameter multiplied by the height. Use d for diameter, and in general use the initial letter for a word in a formula.

7. The volume of a prism is equal to the product of the base and height. Evaluate for $b = 17.5$, $h = 8.5$.

8. The volume of a cylinder is equal to the product of the base and height. Evaluate for $b = 23.2$, $h = 14.3$.

9. The volume of a cylinder divided by the base is equal to the height. Evaluate for $v = 28.8$, $b = 2.4$.

10. The volume of a pyramid is equal to one third the product of the base and height. Evaluate for $b = 16.4$, $h = 8.4$.

11. The volume of a pyramid divided by one third the height is equal to the base. Evaluate for $v = 7.5$, $h = 4.5$.

Exercise 18. Shop Formulas

Examples 1 to 4, oral

1. Given $b = 7$, $h = 9$, what is the value of bh?

2. Given $b = 12$, $h = 20$, what is the value of $\frac{1}{2}bh$?

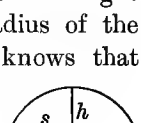
3. Given $r = 10$, what is the value of r^2? of r^3?

4. Given $r = 1$, $\pi = 3\frac{1}{7}$, what is the value of πr^2?

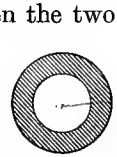
5. A bar of metal has for cross section an equilateral triangle each side of which is s. The area of the cross section is given by the formula $a = \frac{1}{4}s^2\sqrt{3}$. Taking $\sqrt{3}$ as equal to 1.732, find to two decimal places the area of the cross section when $s = 7$.

6. The foreman of a shop knows that the safe load (l) in pounds that can be hoisted by a rope c inches in circumference is given by the formula $l = 100\,c^2$. How many pounds can he safely allow for a rope that is $2\frac{1}{4}$ in. in circumference?

7. A carpenter wishes to put up a circular arch of height h and span $2s$. It is necessary to find the radius of the circle so that he may make his pattern. He knows that $r = \dfrac{s^2 + h^2}{2h}$. Find the radius, given $h = 2$, $s = 4$; given $h = 1$, $s = 1$.



8. The area of a circle with radius a being πa^2, and of one with radius b being πb^2, the area of the ring between the two circumferences is $\pi a^2 - \pi b^2$. What is the area of the ring if $a = 7$, $b = 4$, and $\pi = 3\frac{1}{7}$?



9. A workman needs to find the area of the metal in a cross section of iron pipe the exterior diameter of which is 10 in. and the interior diameter 9 in. Find this from the formula $\pi a^2 - \pi b^2$, where a and b are radii (semidiameters) and $\pi = 3\frac{1}{7}$.

Exercise 19. Review of Chapter I

1. A brick dealer delivered to a contractor the following consignments of bricks: 36 M, 21 M, 45 M, 72 M, 36 M, 19 M, and 27 M. How many bricks were delivered?

2. Find the cost of the bricks in Ex. 1 at \$18 per M.

3. If hemlock timber is worth \$38 per M, find the cost of a stick in which $l = 12$ ft., $w = 8$ in., $t = 5$ in.

4. A ventilating shaft on a school building has a circular cover in which $r = 16$ in. A new copper cover is purchased, the copper weighing 1 lb. per 10 sq. in. Find the weight.

5. Find the interest on \$1250 compounded annually for 3 yr. at 5%.

6. The cost in dollars per foot of sewer pipe of diameter d inches is given by the formula $c = 0.004\,d^2 + 0.14$ in a trade price list. Find the cost of 2400 ft. of 14-inch pipe.

7. A real-estate dealer purchases for a customer a field in the form of a trapezoid of dimensions $b = 30$ rd., $b' = 20$ rd., $h = 12$ rd. Given that an acre is 160 square rods, find the cost of the field at \$400 per acre.

8. How much greater is the compound interest on \$2400 for 4 yr. at 4%, the interest being compounded semiannually, than the simple interest on \$2400 for 4 yr. at 4%?

Taking the letters to stand for the words as given in this chapter, write each of the following statements in words:

9. $a = bh$.
10. $a = \frac{1}{2}bh$.
11. $a = \frac{1}{2}h(b + b')$.
12. $c = 2\pi r$.
13. $c = \pi d$.
14. $a = \pi r^2$.
15. $i = prt$.
16. $a = p(1 + rt)$.
17. $A = p(1 + r)^n$.

18. Find the value of $p(1 + r)^n$ when $p = \$1500$, $r = 0.03$, and $n = 4$.

CHAPTER II

THE EQUATION

36. Balance Sheet. The following is a condensed balance sheet of the affairs of an individual in business:

BALANCE SHEET OF J. P. DUTTON

ASSETS				LIABILITIES			
Cash . . .	3	800	75	Bills payable . .	6	572	
Mdse. . . .	35	500		Personal credits	7	245	
Personal debits	7	400	80	Proprietorship			
Bills receivable	4	726	50	J. P. Dutton .	37	611	05
	51	428	05		51	428	05

It will be seen that the two sides of this sheet *balance*, the totals being equal. Expressed more briefly,

$$\text{Assets} = \text{Liabilities} + \text{Proprietorship},$$

or $$\text{Proprietorship} = \text{Assets} - \text{Liabilities}.$$

If we wish to ascertain the state of a business at any given time, we must find the above three elements, the proprietorship or available resources being found from this *equation.*

The second equation is called the *equation of accountancy,* and upon it depends the whole practice of bookkeeping. The relation expressed by it must hold throughout all the changes of a business, and the student whose training leads him to see what columns to add in order that the sum of the totals may check the total of some other column will increase his power in accountancy.

37. Equation. An expression of equality between two expressions of number is called an *equation.*

For example, $x + 2 = 8$ is an equation. In this equation $x + 2$ is called the *first member* and 8 is called the *second member.*

In this equation x is usually called the *unknown quantity.* It is really a number to be determined. Unknown quantities are usually represented by x, y, or z, or else by initial letters, such as p for pounds or d for dollars.

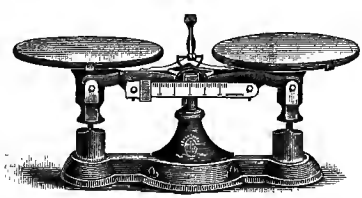
Exercise 20. Using Subtraction

Examples 1 to 8, oral

1. What number increased by 5 is equal to 9?

2. What weight increased by 8 lb. is equal to 13 lb.? In the equation $p + 8 = 13$, what is the value of p?

3. What number increased by 9 is equal to 20? What is the value of x in the equation $x + 9 = 20$?



4. If these scales balance when we place $x + 4$ oz. on one side and 16 oz. on the other side, how much weight will be left on each of the sides if we take 4 oz. from each? What is the value of x?

5. If equals are subtracted from equals, what can be said of the remainders?

Find the value of x in each of the following:

6. $x + 9 = 21$.

7. $x + 19 = 21$.

8. $x + 19 = 49$.

9. $x + 1.7 = 9.1$.

10. $x + 5.7 = 9.2$.

11. $x + 0.7 = 0.9$.

12. $x + 9\frac{1}{2} = 10\frac{1}{8}$.

13. $x + 3\frac{1}{4} = 4\frac{1}{2}$.

14. $x + 5\frac{2}{3} = 7\frac{3}{4}$.

Pages 30–33 consider informally four simple methods of solving equations. The necessary axioms are given on page 34.

Exercise 21. Using Division

Examples 1 to 16, oral

1. If twice the cost price of a table is \$16, what is the cost price? If $2c = \$16$, what is the value of c?

2. If 5 times the cost of an article is \$35, what is the cost? If $5c = \$35$, what is the value of c?

3. At $6¢$ each, how many oranges can I buy for $48¢$? If $6n = 48$, what is the value of n?

4. At $12¢$ a yard, how many yards of ribbon can I buy for $48¢$? If $12r = 48$, what is the value of r?

5. At $40¢$ a dozen, how many dozen eggs can I buy for \$1.60? If $40e = 160$, what is the value of e?

6. If I put equal weights on the two sides of some scales like those shown on page 30, then $\frac{1}{5}$ of the weight on one side will just balance what part of the weight on the other side?

7. If equals are divided by equals, what can be said of the quotients?

Find the value of x in each of the following:

8. $3x = 6$.

9. $3x = 9$.

10. $2x = 6$.

11. $4x = 8$.

12. $5x = 10$.

13. $6x = 24$.

14. $7x = 35$.

15. $9x = 72$.

16. $7x = 91$.

17. A wholesale dealer offers Mr. Michaels 16 tubs of butter containing 52 lb. each for \$325. How much is that per pound to the nearest cent?

18. A man paid \$780 for a certain number of cattle at an average price of \$65 a head. How many did he buy?

19. A dealer paid \$10,325 for some automobiles at an average price of \$1475 each. How many did he buy?

Exercise 22. Using Addition

Examples 1 to 13, oral

1. What must be added to 2 to make 9? to $9-7$ to make 9? to $x-7$ to make x? to $n-7$ to make n?

2. If $x-7=30$, what must be added to these equals to make the first member x? What does the second member then become? What is the value of x?

3. If $n-8=20$, what must be added to these equals to make the first member n? What is the value of n?

4. If $n-6=40$, how will you proceed to find the value of n? What is the value of n?

5. If \$6 is subtracted from the price of a certain article, the result is \$12. What is the price?

6. If equals are added to equals, what can be said of the results?

Find the value of x in each of the following:

7. $x-1=7$.	**14.** $x-27=72$.	**21.** $x-2\frac{1}{2}=3\frac{1}{2}$.
8. $x-2=7$.	**15.** $x-34=62$.	**22.** $x-2\frac{1}{4}=5\frac{3}{4}$.
9. $x-4=6$.	**16.** $x-71=89$.	**23.** $x-6\frac{2}{3}=7\frac{2}{3}$.
10. $x-5=8$.	**17.** $x-8.7=8.7$.	**24.** $x-4\frac{7}{8}=5\frac{3}{4}$.
11. $x-6=9$.	**18.** $x-4.9=5.7$.	**25.** $x-3\frac{4}{5}=8\frac{1}{2}$.
12. $x-7=3$.	**19.** $x-0.78=0.35$.	**26.** $x-6\frac{2}{3}=4\frac{1}{4}$.
13. $x-8=5$.	**20.** $x-0.87=0.63$.	**27.** $x-9\frac{3}{4}=8\frac{3}{5}$.

28. Joseph Ellis has paid me \$125 on account and still owes me \$84.75. How much did he owe me at first?

29. A coal company reduced its surplus \$128,000 this year to help make up its usual dividend. Its surplus is now \$3,400,000. How much was it last year?

Exercise 23. Using Multiplication

Examples 1 to 12, oral

1. By what must $\frac{1}{4}$ in. be multiplied to make 1 in.?

2. By what must $\frac{1}{4}x$ be multiplied to make x?

3. By what must $\frac{3}{4}$ be multiplied to make 3?

4. By what must $\frac{3}{4}x$ be multiplied to make $3x$?

5. By what must both members of the equation $\frac{x}{4} = 7$ be multiplied to give the value of x? What is the value of x?

6. If $\frac{x}{3} = 9$, what is the value of x? Prove it.

7. If equals are multiplied by equals, what can be said of the products?

Find the value of x in each of the following:

8. $\frac{1}{2}x = 9$.

9. $\frac{x}{3} = 7$.

10. $\frac{x}{4} = 8$.

11. $\frac{x}{5} = 5$.

12. $\frac{x}{6} = 4$.

13. $\frac{1}{4}x = 49$.

14. $\frac{x}{17} = 16$.

15. $\frac{x}{13} = 14$.

16. $\frac{x}{15} = 12\frac{1}{3}$.

17. $\frac{x}{16} = 14\frac{3}{4}$.

18. $\frac{1}{16}x = 26$.

19. $\frac{x}{2.7} = 32$.

20. $\frac{x}{4.1} = 3.9$.

21. $\frac{x}{5.2} = 1.7$.

22. $\frac{x}{0.61} = 0.75$.

23. If $\frac{1}{32}$ of the thickness of a steel plate is $\frac{1}{8}$ in., what is the thickness of the plate?

24. If $\frac{1}{48}$ of the diameter of a steel rod is $\frac{1}{96}$ in., what is the diameter of the rod?

25. If 1% of the cost of a house is \$47.50, what is the cost of the house?

38. Axioms. Statements like the following, admitted to be true without proof, are called *axioms.*

1. *If equals are added to equals, the results are equal.*

For example,

	$7 = 7$		$6 - 2 = 4$		$3x - 5 = 7$
Adding	$2 = 2$	Adding	$2 = 2$	Adding	$5 = 5$
we have	$9 = 9$	we have	$6 = 6$	we have	$3x = 12$

2. *If equals are subtracted from equals, the results are equal.*

For example,

	$7 = 7$		$9 + 4 = 13$		$2x + 3 = 9$
Subtracting	$2 = 2$	Subtracting	$4 = 4$	Subtracting	$3 = 3$
we have	$5 = 5$	we have	$9 = 9$	we have	$2x = 6$

3. *If equals are multiplied by equals, the results are equal.*

For example,

	$7 = 7$		$3 + 2 = 5$		$\frac{1}{2}x + 3 = 7$
Multiplying	$2 = 2$	Multiplying	$2 = 2$	Multiplying	$2 = 2$
we have	$14 = 14$	we have	$6 + 4 = 10$	we have	$x + 6 = 14$

4. *If equals are divided by equals, the results are equal.*

For example,

	$7 = 7$		$3 + 2 = 5$		$3x + 6 = 15$
Dividing	$2 = 2$	Dividing	$2 = 2$	Dividing	$3 = 3$
we have	$\frac{7}{2} = \frac{7}{2}$	we have	$\frac{3}{2} + 1 = \frac{5}{2}$	we have	$x + 2 = 5$

5. *Like powers or like roots of equal numbers are equal.*

For example,

$3 = 3$	$27 = 27$	$x = 2$	$x^2 = 25$
$\therefore 3^2 = 3^2$	$\therefore \sqrt[3]{27} = \sqrt[3]{27}$	$\therefore x^2 = 4$	$\therefore x = \sqrt{25}$
or $9 = 9$	or $3 = 3$	and $x^3 = 8$	or $x = 5$

6. *Numbers equal to the same number are equal to each other.*

If John's money = \$5	If $x + 5000$ = A's share
and Henry's money = \$5,	and 7000 = A's share,
then John's money = Henry's money.	then $x + 5000 = 7000$.

39. Applications. The application of the equation to commercial problems is best understood from a few illustrations.

1. The number 20 is 10% of what other number?

From the statement 20 is 10% of what number,

we have $$20 = 0.10\,x.$$

That is, the word "is" becomes the symbol =; the word "of" becomes the absence of a sign, and this absence signifies multiplication; and the words "what number" become the symbol x. The equation is easily solved by dividing each of the equals by 0.10.

2. A's salary and his share of profits together amount to \$2400. If his salary is \$1500, how much is his profit?

From the statement that salary and profits amount to \$2400,

we have $$1500 + x = 2400.$$

This equation is easily solved by subtracting 1500 from each member.

Exercise 24. Problems

Express the following statements as equations, and solve:

1. The double of A's money is \$9000.

2. Half of B's money is \$1875.

3. The double of C's money, together with \$30, is \$900.

4. The number 80 is what per cent of 960?

5. The number 75 is 5% of what number?

6. The number 940 is what per cent of 7250?

7. A man's assets are \$17,275 and his liabilities are \$4750. Find his proprietorship.

8. If A's money is decreased by \$750, it becomes \$1725. How much money has A?

9. If twice B's money is increased by \$42, it becomes \$684. How much money has B?

40. Further Applications. 1. A's share of the profits of a business is twice B's share, and together their profits are \$9000. Find the profit of each.

If we let x = the number of dollars in B's share,

then we must let $2x$ = the number of dollars in A's share.

Then $x + 2x = 9000$,

or $3x = 9000$;

whence $x = 3000$.

That is, B's share is \$3000, and therefore A's share is \$6000.

2. A and B are partners, A furnishing a building worth \$1200 a year and also supplying \$35,000 of capital, and B supplying \$27,000 of capital. If \$32,200 is available to cover the profit and the rent, find each man's share of this fund.

Since every \$1000 of capital should receive the same profit, we may let

$35x$ = the number of dollars in A's return on his capital,

and $27x$ = the number of dollars in B's return on his capital.

Since B's profit + A's profit + A's rent = total amount divided,

we have $27x + 35x + 1200 = 32{,}200$.

Hence we have $62x + 1200 = 32{,}200$,

or $62x = 31{,}000$;

whence $x = 500$.

Therefore $27x = 13{,}500$

and $35x + 1200 = 18{,}700$.

That is, B's share is \$13,500, and A's is \$18,700.

3. For a crop of 450 bu. of apples packed in barrels A received an offer of \$600. The barrels cost 30¢ each, and each barrel contained 2 bu. What was the net offer per bushel?

Since value of apples + value of barrels = \$600, total value,

we have $450x + 67.50 = 600$.

Exercise 25. Problems

1. Complete the solution of Ex. 3 on page 36.

2. A's money is twice B's, and the sum of A's and B's money is \$4800. How much money has each?

3. B's money is twice A's, and the difference between A's and B's money is \$800. How much money has each?

4. The number 90 is what per cent of 720?

5. A standard gold coin is 9 parts pure gold and 1 part alloy. What per cent of the coin is pure gold?

6. A's share of a profit of \$7200 is three times B's. Find the share of each.

7. A owns 31 of the 50 shares of stock in a business, B owning the rest. If a profit of \$7500 is to be divided between them, how much should each receive?

8. A, B, and C own respectively 21, 17, and 2 shares of the 40 shares of stock in a business. How should they divide a profit of \$8400?

9. There are 560 shares of stock in a business, of which A owns 190 shares and B owns the rest. B furnishes a building which is worth \$1800 a year. If the profits and rent are together \$15,800 this year, how should this sum be divided?

10. Divide \$900 among three partners, A, B, and C, so that A shall receive twice as much as B, and B shall receive three times as much as C.

11. A farmer receives a bid of \$700 f.o.b. New York for a potato crop of 700 bu. If the freight charges are 10¢ a bushel, how much does the farmer receive per bushel?

The letters f.o.b. mean "free on board"; that is, the farmer is to deliver the potatoes on board the train at New York with all transportation charges paid.

41. Problems relating to Per Cents. The following problems are typical of those met in percentage.

1. A man receives 6% on some money invested and adds \$60 to the amount received, thus making \$300. How much has he invested?

Let x = the number of dollars invested.

Then $0.06\,x$ = the number of dollars received,

and $0.06\,x + 60$ = the number of dollars stated, or 300.

$\therefore\ 0.06\,x + 60 = 300.$

Subtracting 60, $0.06\,x = 240.$

Dividing by 0.06, $x = 4000.$

Therefore the man has \$4000 invested.

Check. 6% of \$4000 = \$240, and \$240 + \$60 = \$300.

It should be observed that we do not let x equal the *money*, but we let x = the *number* of dollars. Then when we find that $x = 4000$, we know that this is the *number* of dollars, and that \$4000 is the sum invested.

2. What per cent above cost must a man mark his goods so as to allow a discount of 20% and still make a profit of 20%?

The question is to find what per cent the marked price is of the cost, and then to find how much this is above cost.

Let c = the number of dollars of cost.

Then $1.20\,c$ = the number of dollars of selling price.

Let m = the number of dollars of marked price.

Then $0.80\,m$ = the number of dollars of selling price.

Therefore $0.80\,m = 1.20\,c.$

Dividing by 0.80, $m = 1.50\,c.$

That is, the goods must be marked 50% above cost.

Check. If from $1.50\,c$ we take 20% of it, we have left 80% of $1.50\,c$, or $1.20\,c$. This $1.20\,c$ is 20% more than c.

For example, if the goods cost the dealer \$80, he must mark them at $1.50 \times \$80$, or \$120. The selling price is then 20% less, or \$96, and this is 20% more than \$80.

Exercise 26. Problems relating to Per Cents

Examples 1 to 12, oral

1. The number 6 is 6% of what number?
2. The number 3 is 6% of what number?
3. The number 6 is 3% of what number?

Solve the following equations:

4. $6\%\, x = 18$.
5. $5\%\, x = 15$.
6. $4\%\, x = 12$.
7. $3\%\, x = 9$.
8. $2\%\, x = 6$.
9. $12\%\, x = 24$.
10. $x + 6\%\, x = 106$.
11. $x + 5\%\, x = 105$.
12. $x + 5\%\, x = 210$.

13. If 18% by weight of wheat is lost in grinding it into flour, how much wheat is used if the loss is 360 lb.?

14. From Ex. 13 how many pounds of wheat are needed to produce 820 lb. of flour?

15. From Ex. 13 how many pounds of wheat are needed to produce 779 lb. of flour?

16. A dealer sells for \$32.20 a suit of clothes which cost him \$28. What is his per cent of gain on the cost?

17. A suit of clothes was sold for \$30.80 after a discount of 12% was allowed on the marked price. What was the marked price?

18. The profits on a business this year are \$6344, or 22% more than they were last year. What were they last year?

19. Water in freezing expands 9% of its volume. How many cubic inches of water will make 501.4 cu. in. of ice? How many gallons (231 cu. in.) of water are needed?

20. At what per cent above cost must a dealer mark his goods so that he can deduct 25% and make a profit of 25%? What will be the marked price if the goods cost \$800?

21. The experience of the Markham Grocery Co. shows that 15% of the sales are bad debts. At what price must it sell an article costing \$3 so as to make a profit of 25% on the cost and allow for the bad-debt factor?

The word "factor" is commonly used in this sense.

22. The Lesher Company bought broadcloth at \$2.80 a yard, and wishes to mark it so as to allow the cash customers 5% discount and still realize a gain of 20% on the cost. At what price per yard must the broadcloth be marked?

23. A bankrupt firm having assets of \$27,500 and liabilities of \$36,000 will pay what per cent of its indebtedness?

24. A firm of coffee roasters receives an order for 600 lb. of roasted Java coffee. How many pounds of green Java must it roast if the process of roasting causes the green coffee to lose 12% of its weight?

25. What should be the catalogue price of a library table costing \$25 in order to insure a gain of 25% on the cost and still allow a discount of 20% to the purchaser?

26. Farmer Jones has 800 bu. of winter wheat and 300 bu. of spring wheat. An agent tells him that the spring wheat is worth 20% more per bushel than the winter wheat, and offers to take the entire crop for \$2000. Assuming the agent's estimate to be correct, what price per bushel, to the nearest cent, is he offering Mr. Jones for each kind of wheat?

27. The A and B Company estimates that 15% of its receipts covers the cost of doing business, and 10% of its receipts represents profits. How much must it receive for a sprayer that cost \$3.40 if it is to be sold on the above basis?

28. If an article cost \$5 and is to be sold so that 22% of the price received shall cover other items than cost, find the selling price.

Exercise 27. Review of Chapter II

1. My accounts show that J. C. Ford owes me \$275. He agrees to pay me \$25 a month until the amount is paid. How many payments must he make?

2. A buyer offers a manufacturer \$1950 f.o.b. the buyer's town for 16,000 yd. of grade A muslin. The manufacturer estimates the freight at \$30. How much does the offer yield him per yard?

Solve the following equations:

3. $3x = 16.2$.
4. $x - 5 = 11$.
5. $x + 4 = 22.3$.
6. $\frac{x}{3} + 4 = 11$.
7. $2x - 3 = 45$.
8. $1500x + 21 = 4521$.
9. $2\frac{1}{2}x = 17\frac{1}{2}$.
10. $3x - 7\frac{1}{2} = 16\frac{1}{2}$.
11. $5x + 6.1 = 19.6$.
12. $\frac{3x}{4} + 5 = 6\frac{1}{2}$.
13. $7x - \frac{3}{4} = x + 12\frac{3}{4}$.
14. $28.4x + 3x = 62.8$.

15. A and B in their articles of partnership agree that A shall have a salary of \$2500 for managing the business, and the balance of profits shall be divided so that A's share shall be $\frac{2}{3}$ of B's share. Find each man's share of a total of \$4500 available for salary and profit.

16. From C's balance sheet we learn that his assets are \$75,000, and his proprietorship \$40,500. Find his liabilities.

17. Four men, A, B, C, and D, own respectively 27, 35, 42, and 64 shares of a business of which they are the sole owners. If they divide \$84,000 profits, find the amount received by each.

18. If $\frac{2}{3}l = 350$, find l. What axiom or axioms are used?

19. \$35,000 is 7% of what sum of money?

Exercise 28. Review of Chapters I and II

1. A triangular lot in which $b = 85$ ft. and $h = 60$ ft. is offered for sale at \$3 per square foot. Find the asking price.

2. A brick wall, whose front is a rectangle in which $b = 30$ ft. and $h = 20$ ft., is estimated to require 22 bricks for each square foot. How many bricks will it take, and what will be the cost of these bricks at \$27 per M?

3. Find the compound interest on \$900 for 5 yr. at 4%, the interest being compounded annually.

4. Find the value of $p(1+r)^n$ if $p = 8000$, $r = 0.04$, $n = 4$.

5. Find the value of $\frac{1}{2}gt^2$ if $g = 32.2$, $t = 7$.

6. Find the value of $\frac{1}{2}h(b+b')$ if $h = 40$, $b = 80$, $b' = 70$.

7. Find the value of $3x^2y^3 + 2xy^2$ if $x = 3$, $y = 2$.

8. A peach crop of 942 baskets is sold by the grower for \$500. If the baskets cost $10\,¢$ each, how much will he realize per basket on the peaches, to the nearest cent?

Write a formula for every such case.

9. If $2x + 5 = 11$, find the value of x.

10. A circular room in a public building has a diameter of 100 ft. Find the area of the floor.

11. Find each man's share of a \$5000 profit if A's share is twice B's share and B's share is twice C's.

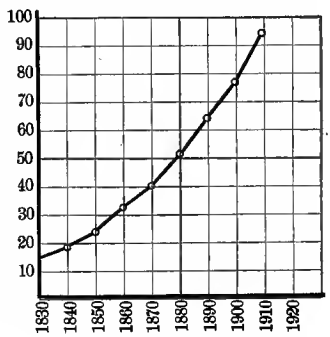
12. A druggist used 11 oz. of alcohol in preparing a 10% solution. How many ounces in the entire solution?

13. A salesman offers to sell Mr. A for \$611.25, f.o.b. Mr. A's station, 300 cans of best white lead, each can containing 10 lb. A dozen cans are packed in each box. Each box costs $20\,¢$ and the total freight is \$6.25. How much is the salesman charging Mr. A per pound for the white lead?

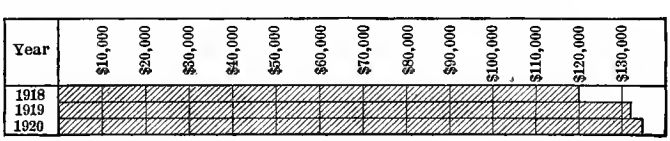
CHAPTER III

ELEMENTARY GRAPHS

42. Appeal to the Eye. In modern commercial life it is found that there is a great advantage in appealing to the eye in statistical matters. We see this in the newspapers and magazines, so that the simpler forms of *graphs* are already familiar to you. As an example of these simpler forms we may consider this graph, which shows the growth in population of the United States from 1830 to 1910, the numbers at the left indicating the millions of population. Such a graph shows much more clearly than the figures the rapidity of increase every ten years.



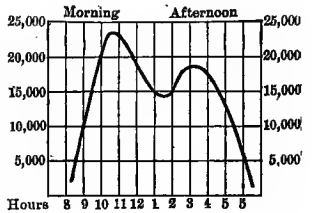
Similarly, the following graph shows at a glance that the rate of growth of sales of a business house from 1918 to 1919 was not maintained in the following year.



Such graphs as those shown above are often called *pictograms.* The second one is called a *bar pictogram* because the numbers in the statistics are represented by bars.

Exercise 29. Pictograms

1. This graph shows by hours the number of calls which pass through one of the larger telephone exchanges handling Wall Street business in New York. Write a statement of what the graph tells you as to peaks of business, and explain the cause of the depression.



2. The following graph shows the value of iron and steel products of several states in millions of dollars in a certain year. Write a statement of the four most important facts which the graph tells you.

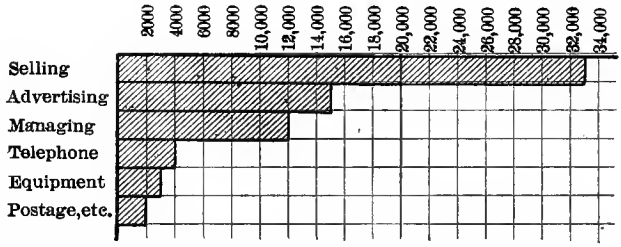


3. Represent the following statistics graphically: The number of national banks in the United States in 1910 was 6996; in 1911, 7163; in 1912, 7307; in 1913, 7404; in 1914, 7473; in 1915, 7560; in 1916, 7578.

4. The total exchanges of clearing houses of our cities for certain years, stated in billions of dollars, were as follows: in 1911, 159; in 1912, 169; in 1913, 174; in 1914, 164; in 1915, 163; in 1916, 241. Represent these facts graphically.

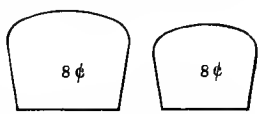
5. The value of gold mined in the United States for certain years, stated in millions of dollars, was as follows: in 1910, 96; in 1911, 97; in 1912, 93; in 1913, 89; in 1914, 95; in 1915, 101; in 1916, 92. Represent these facts graphically.

43. Value to Directors. An example of the value of the pictogram is seen in this summary of the report of expenses of a certain establishment. On preceding years the figures

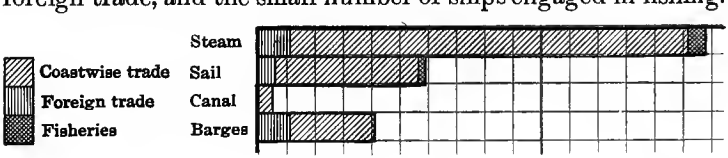


had been presented without exciting any comment, but when the treasurer set forth the facts by means of this graph, the directors saw at once that the amount allotted to advertising was too small in proportion to that given to management.

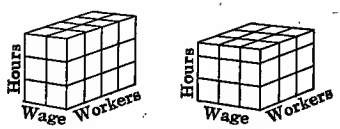
44. Value in Advertising. Advertisers are very quick to see the value of the appeal to the eye. A baker advertised without much effect that he sold a 16-ounce loaf for the same price that his rival charged for a 12-ounce loaf; but when he resorted to the graphic method here shown, the effect was instantaneous.



45. Multiple Pictograms. In pictograms relating to statistics the bar diagram is often used to compare several features at once. For example, this pictogram shows the number of American ships afloat in a certain year before we began extensive building, each square indicating 1000. We see at once the great preponderance of our coastwise trade over our foreign trade, and the small number of ships engaged in fishing.

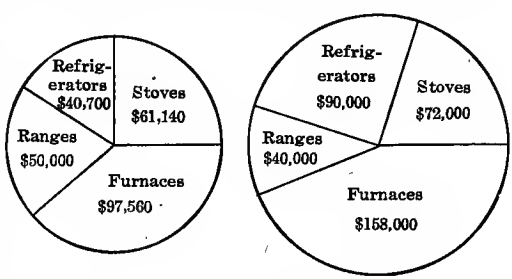


46. Solid Pictogram. It is sometimes convenient to represent a solid by means of a pictogram, for the purpose of showing three related sets of statistics. Thus to represent graphically the total wages paid in the clothing industry of two cities, as well as to compare the wages per hour, the hours per week, and the total number of workers, we may make use of a rectangular solid. Here the number of workers is shown in one dimension, the wage per hour in another, and the number of hours per week in another. We see at once that city A has $\frac{4}{3}$ as many workers in this industry as city B, that the workers in A are paid only $\frac{2}{3}$ as much per hour, and that the number of hours per week in A is $\frac{6}{5}$ as many as in B. Furthermore, the total weekly wages are represented by the volumes of the blocks, which are respectively 24 and $22\frac{1}{2}$.



CITY A CITY B

47. Circular Pictogram. The circle is a favorite means of comparing statistics. For example, these circles represent the comparative business that has been done by a certain house in two consecutive years. We see at a glance that the sales of refrigerators and furnaces have increased in relative importance, and that, although there is an increase from \$61,140 to \$72,000 in the sales of stoves, there is a relative shrinkage. The sale of stoves has dropped from the position of about $\frac{1}{4}$ the total business to $\frac{1}{5}$, and the sale of ranges has shrunk both absolutely and relatively.



Exercise 30. Pictograms

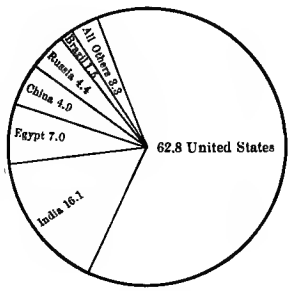
1. Using bars of the same width to represent the number of million bushels of wheat received in New York in five consecutive years, construct a bar diagram to set forth these facts: first year, 98; second, 119; third, 130; fourth, 140; fifth, 159.

In this case it is convenient to use squared paper, which can be purchased at any stationer's, representing 10 by one space. The use of squared paper in more advanced work in algebra is explained later.

2. Using bars of the same width to represent the number of wage earners receiving the salaries mentioned below in the manufacturing industries of this country in a certain year, construct a bar diagram to set forth the facts approximately. Divide each bar to represent the proportion of men, women, and children in each class, shading the divisions differently.

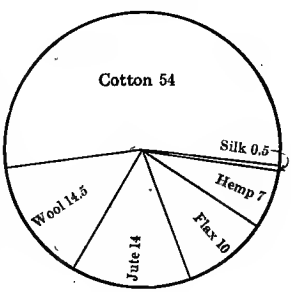
Weekly Earnings	Men 16 and Over	Women 16 and Over	Children	Total
Under $3	92,535	77,826	55,432	225,793
$3–$4	96,569	115,741	52,316	264,626
$4–$5	149,531	158,926	31,656	340,113
$5–$6	177,550	173,713	12,430	363,693
$6–$7	272,288	176,224	5,773	454,285
$7–$8	327,726	124,061	1,416	453,203
$8–$9	336,669	86,467	553	423,689
$9–$10	557,046	62,193	226	619,465
$10–$12	654,435	54,340	83	708,858
$12–$15	714,816	26,207	13	741,036
$15–$20	609,797	8,516	1	618,314
$20–$25	170,571	1,273	. . .	171,844
$25–over	85,005	397	. . .	85,402
Total	4,244,538	1,065,884	159,899	5,470,321

3. This pictogram shows the percentage of the world's mill supply of cotton contributed by each country in a recent year. Write the statement in statistical form.



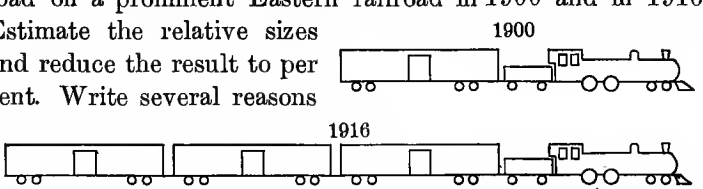
4. In Ex. 3 write a statement as to four important facts which the graph would tell you even if the per cents were omitted. For example, about how would you rank the United States in comparison with India or with Egypt?

5. This pictogram shows the relative quantities of the six most important of the world's textile fibers. Write a statement as to four important facts which the graph tells you.



6. From Ex. 5 find the per cent which the quantity of flax is of the quantity of jute; of the quantity of hemp.

7. This pictogram shows the comparative sizes of a trainload on a prominent Eastern railroad in 1900 and in 1916. Estimate the relative sizes and reduce the result to per cent. Write several reasons



to explain why the size of the trainload has increased.

8. Draw a pictogram in any form showing the relative space given in this book to the first three chapters.

9. The Metropolitan Sewerage Commission in its report gives the following as the number of bacteria per cubic centimeter in the waters near New York: Atlantic Ocean, 120; Lower New York Bay, 1600; Harlem River, 15,600. Draw a pictogram representing these facts, using three circles of the same size, the relative number of bacteria being shown by dots, one dot being used to represent 120 bacteria.

10. Representing total acreage either by circles or by rectangles with equal bases but different altitudes, show by a pictogram the number of acres of land of the United States devoted to crops of various kinds, and then estimate from the pictogram the relative sizes of the various crops in each of the two years represented in the following table of statistics:

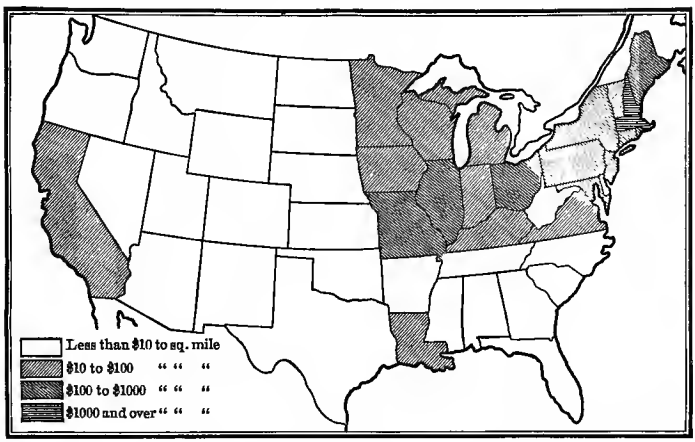
Crops	First Year	Second Year
1. Cereals (corn, wheat, etc.) . .	184,982,220	199,395,963
2. Other grains and seeds . . .	4,075,120	5,157,374
3. Tobacco, hay, and cotton . .	87,067,630	105,619,525
4. Sugar crops	790,308	1,285,031
5. Minor field crops	286,213	390,784
6. Vegetables	5,628,220	7,073,379
7. Small fruits (strawberries, etc.)	309,770	272,460
8. Flowers and nursery products	68,799	98,866
	283,208,280	319,293,382

11. In Ex. 10 find from the graphs what crop used a smaller acreage in the second year than in the first year.

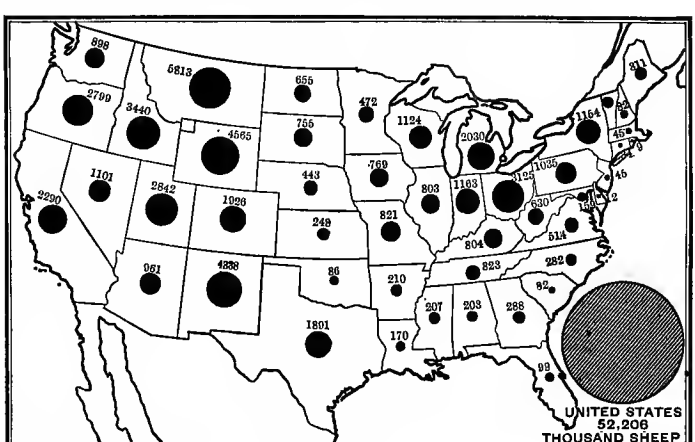
12. How does the relative increase in acreage of the sugar crop compare with that of vegetables? with that of tobacco, hay, and cotton? with that of flowers and nursery products?

It is desirable to take the latest available statistics and to consider what causes bring about the changes.

48. Cartogram. This map, or *cartogram*, shows the distribution of the manufacture of boots and shoes in this country.



The following cartogram shows the number of thousands of sheep in the United States in a certain year.



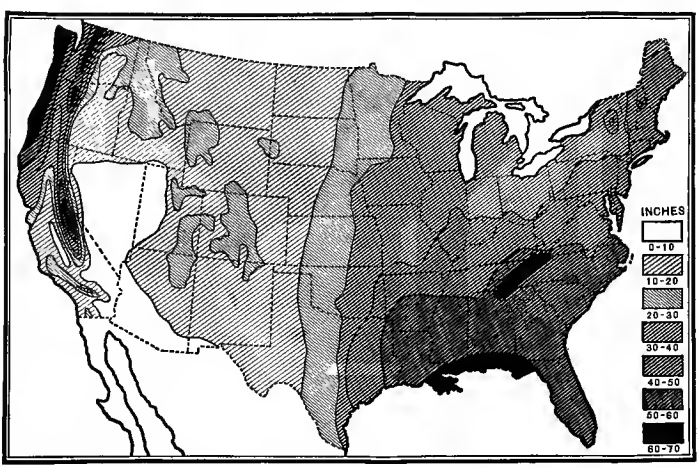
Exercise 31. Cartograms

1. Using the first map on page 50, write the names of the states in which the manufacture of boots and shoes was $1000 and over per square mile.

2. Give a reason for the concentration of boot and shoe manufacture in the states indicated in Ex. 1.

3. From page 50 write in order of number of sheep the names of the ten states producing the largest number.

4. Write a statement in the way of a report to a shipping house as to the mean annual rainfall in inches, containing four important facts and based on the following cartogram:



Such information is of great commercial importance, particularly to houses that sell goods which deteriorate quickly in a damp climate.

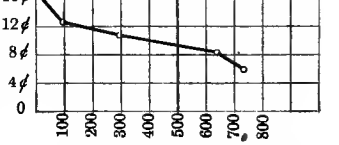
5. Sketch a map of the city or village in which you live, and shade it to show approximately the area occupied chiefly by stores, that occupied chiefly by manufacturing plants, and that occupied chiefly by residences.

49. Graphs of Functions. In business it is recognized that the demand for an article changes with the price, that the grocery trade in a neighborhood changes with the change in population, and that in normal times the price of wheat changes with the amount which is grown. In each case one number varies when another one does; that is, one number depends upon another number for its value.

If a number x depends upon another number y for its value, x is called a *function* of y. The study of functions of one kind or another is the central feature of all mathematics in general and of commercial algebra in particular, and the student should be perfectly familiar with their significance.

Thus, since $c = 2\pi r$, the circumference is a function of the radius, and similarly in all the formulas of algebra. Likewise, since $i = prt$, we see that the interest is a function of the time when the principal and rate are fixed; a function of the rate when the principal and time are fixed; and a function of the principal when the rate and time are fixed.

The functional relation can often be represented graphically. For example, it was found in a certain market that when strawberries sold at 16¢ a quart, 25 qt. were sold daily; when the price dropped to 12¢, 100 qt. were sold daily; at 10¢, 300 qt. were sold, and so on, the average number of quarts sold per day being a function of the price. The relation of price to sales may be indicated graphically as here shown, and the story is told much more vividly than by a mere recital of figures.



Such a graph is known as a *demand curve*, and commercial houses often find that a study of these curves is profitable in making their plans for purchasing stock. The axes need not be specially drawn on the coordinate paper unless desired.

Exercise 32. Graphs of Functions

1. The total annual sales of a grocery for a period of ten years were as follows: \$16,000, \$18,000, \$20,000, \$25,000, \$24,000, \$26,000, \$28,000, \$36,000, \$43,000, \$52,000. Draw a graph to show the trend of business.

2. From the graph in Ex. 1 estimate the probable amount of business the next year. Will it be advisable for the store to enlarge its plant and stock in this neighborhood?

3. Construct a demand curve based on the following facts which have been ascertained with respect to a certain article:

Price . .	12¢	10¢	8¢	6¢	4¢	2¢
No. sold .	400	500	1800	3000	8000	8200

4. The following table shows the population in millions of the United States in various years, and also the density of population, that is, the number of persons per square mile:

Year	Population	Density	Year	Population	Density
1800	5	6.5	1860	31	10.6
1810	7	4.3	1870	39	13.0
1820	10	5.5	1880	50	16.9
1830	13	7.3	1890	63	21.2
1840	17	9.7	1900	76	25.6
1850	23	7.9	1910	92	31.0

Draw both graphs (population and density) on the same sheet, representing the years on the horizontal line and the population and density on the vertical line.

For population let one space represent 10 millions, and for density of population per square mile let one space represent 10 persons.

5. Since the density of population is found by dividing the population by the number of square miles, we have $D = P/A$. From this show that $P = AD$, and $A = P/D$. From Ex. 4 determine A for 1840 and 1850, and find whether there was any change in area in this period.

6. A grocer's records of sales of peaches for a certain year showed that at \$1.25 a basket he sold 10 baskets; at \$1.00, 30 baskets; at 75¢, 70 baskets; at 60¢, 85 baskets; and at 50¢, 90 baskets. Draw the demand curve and estimate the probable demand when the price is 90¢ a basket.

7. If the business of the grocer mentioned in Ex. 6 has increased 20% since last year, when the demand facts were recorded, draw a demand curve for this year, assuming that he will sell 20% more at each price mentioned above than he did last year. Using your curve as a guide, estimate how many baskets of peaches he will probably be able to sell this year at 70¢.

8. Consult the last census report on manufactures in the United States, and divide the states into classes as to the value of their manufactures. Make a cartogram of the United States, showing the value of manufactures in the various states, shading equally the states that fall in the same class.

9. In Ex. 8 explain why the heaviest output of manufactured goods is found in the states which were so shaded as to show the greatest value of manufactures.

10. Find if possible, say from an encyclopedia, the population of Russia, the United States, Germany, and France for each decade from 1810 to 1910, and draw graphs on the same sheet showing the increase or decrease of population in each country. By extending the United States graph, continuing its direction in 1910, estimate the probable population of our country in 1920 and in 1930, neglecting war conditions.

Exercise 33. Review of Chapter III

1. The average prices per pound of lake copper in five recent years have been as follows: 12.7¢, 16.7¢, 16.7¢, 13.5¢, 28¢. Draw a bar diagram representing these prices.

2. Draw a bar diagram showing the disposition of the net income of a large Western railroad for a recent year, using first a separate bar for each item, and then using a single bar to represent the total, dividing it into proportional parts to represent the separate items. The figures are as follows, but only approximations should be used:

Sinking-fund requirements	$411,800
Dividends on preferred stock . . .	3,981,740
Dividends on common stock . . .	17,783,328
Additions and betterments	1,083,459
Surplus to profit-and-loss account . .	5,544,032

3. The total sales of a certain shoe dealer for a series of years were, respectively, $12,000, $15,000, $18,000, $19,000, $22,000, $30,000, and $32,000. The census of the ward in which the business was located showed the following population for the same years: 3200, 4100, 4900, 5200, 6000, 6400, and 6800. Draw on the same diagram graphs of the increase of business and the increase of population of the neighborhood for these years. Did the increase of the business keep pace with the rate of increase of the population throughout the period? Compare the two movements throughout the period.

4. Construct the graph of total daily attendance in your class for last week, based on the records.

5. Secure from a dealer in your neighborhood the record of sales of some article for which there is a seasonal demand. Construct the demand curve.

Exercise 34. Review of Chapters I–III

1. The space s traversed in t units of time by a body moving at a uniform velocity v per unit of time is expressed by the formula $s = vt$. By substituting the particular values here given, solve for the third quantity in each case: $v = 30$ ft. per second, $t = 5$ sec.; $s = 600$ ft., $v = 10$ ft. per second; $s = 300$ meters, $t = 30$ sec.

2. Solve the equation $3x + 7 = 22$.

3. If x is a positive integer greater than 1, find by substituting several different numbers for x which expression is greater, $x^2 + 2x$ or $x^3 + 1$.

4. Find the value of $(1 + r)^n$ when $r = 0.03$ and $n = 4$.

5. A merchant desiring to build a branch store near the residential section of a certain city discovers two sites equally well located and costing the same amount. One is a square plot 65 ft. by 65 ft.; the other is a trapezoid in which $b = 80$ ft., $b' = 60$ ft., and $h = 65$ ft. Which plot will give him the greater floor space for his money?

6. Five competing lumber companies, A, B, C, D, and E, formed a pool and established a combined selling agency. The equalizer was instructed to assign orders to the member firms in proportion to the average sales of each for the five-year period preceding the formation of the pool. These averages were respectively: A, \$120,000; B, \$150,000; C, \$200,000; D, \$250,000; E, \$300,000. What amount of business should he assign to each company if he sold for the pool \$1,632,000 worth of lumber?

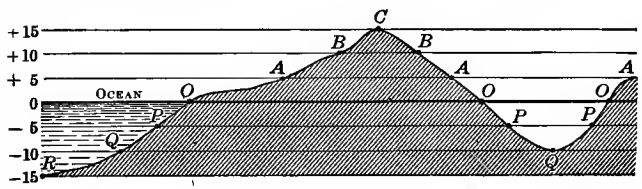
7. What is the value of $a + 2b + 4c$ when $a = 52$, $b = 50$, and $c = 12$?

8. Draw a cartogram of your city or village and indicate by dots the general distribution of the retail stores.

CHAPTER IV

NEGATIVE NUMBERS

50. Directed Numbers. In certain surveys the level of the ocean is taken as the *level of reference*. Distances above this level are called *positive elevations*, and distances below it are called *negative elevations*. The numbers representing these distances may be considered as counted in opposite directions from the level of reference, and hence they are known as *directed numbers*. The following diagram represents the cross section of some land near the ocean, with elevations in feet:



The elevation above sea level of the points marked A is + 5 ft., read "plus five feet," and the elevation of the points marked Q is − 10 ft., read "minus ten feet." Similarly, the elevation of the point R is − 15 ft.

Exercise 35. Directed Numbers

All work oral

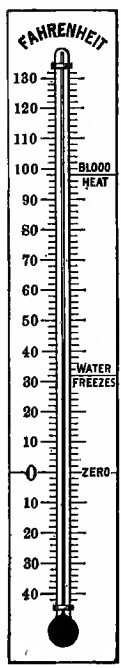
From the above diagram state the difference in level between the following points:

1. O, A.	**3.** B, C.	**5.** O, B.	**7.** P, O.	**9.** R, O.
2. A, B.	**4.** A, C.	**6.** O, C.	**8.** Q, O.	**10.** P, Q.

51. Directed Numbers on the Thermometer. When it is necessary to distinguish between temperature below zero and temperature above zero, we write $+10°$ for 10° above zero and $-10°$ for 10° below zero.

If the temperature is 20° above zero and it decreases 15°, it is then 5° above zero, or + 5°. If it decreases 5° more, it is then 0°. If it decreases 5° more, it is 5° below zero, or − 5°. If it decreases 20° more, it is then 25° below zero, or − 25°.

52. Two Uses for Signs. We therefore find a new meaning for the signs + and −. They not only are used to indicate addition and subtraction (signs of *operation*) but they tell on which side of zero a number is situated (signs of *quality*).



53. Positive and Negative Numbers. The ordinary numbers which we use in arithmetic are called *positive numbers.*

Thus 3°, 3 in., $\frac{3}{4}$, $\sqrt{3}$, are all positive numbers. If we wish to make this fact emphatic, we may write them thus: + 3°, + 3 in., + $\frac{3}{4}$, + $\sqrt{3}$, but otherwise the + sign is not necessary here. The expression + 3 is read "positive 3" or "plus 3."

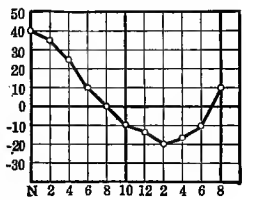
Numbers on the other side of zero from positive numbers are called *negative numbers.*

Thus − 3° is a negative number. If distance eastward from a certain village is called positive, distance westward from the village is called negative, so that we may have + 10 mi. and − 10 mi. The expression − 3 is read "negative 3" or "minus 3." We may think of zero as being either positive or negative, since it divides the two classes of numbers.

54. Absolute Value. The numerical value of a quantity, without reference to its sign, is called its *absolute value.*

The absolute value of − 4 is 4, and that of $-a$ is a.

55. Curve Tracing with Negative Numbers. If the temperature in St. Paul in the winter falls from $+40°$ at noon to $-20°$ at 2 A.M., and then rises again to $+10°$ at 8 A.M., as here shown, the curve tells us that it was below zero from 8 P.M. until about 7 A.M.



In this case we represent negative numbers below the horizontal line, which represents zero.

Exercise 36. Curve Tracing

Examples 1 to 6, oral

1. By the above curve, when was the temperature 35°?
2. At what time was it $-10°$? $+10°$? $+5°$? $-5°$?
3. What was the temperature at 4 A.M.? at 8 A.M.?
4. At what time did the temperature cease falling?
5. How much did it rise from 2 A.M. to 6 A.M.? to 8 A.M.?
6. What was the difference between the temperature at 6 P.M. and 10 P.M.? 8 A.M. and 2 A.M.? 10 P.M. and 2 A.M.?

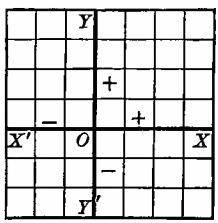
Trace the curves to show the following variations in temperature for the twenty-four hours:

7. Noon, 23°; 2 P.M., 25°; 4 P.M., 18°; 6 P.M., 7°; 8 P.M., 0°; 10 P.M., $-4°$; midnight, $-8°$; 2 A.M., $-10°$; 4 A.M., $-5°$; 6 A.M., 0°; 8 A.M., 12°; 10 A.M., 16°; noon, 24°.

8. Noon, 28°; 2 P.M., 25°; 4 P.M., 16°; 6 P.M., 5°; 8 P.M., 0°; 10 P.M., $-5°$; midnight, $-8°$; 2 A.M., $-5°$; 4 A.M., 0°; 6 A.M., 3°; 8 A.M., 8°; 10 A.M., 17°; noon, 22°.

9. Noon, 0°; 2 P.M., 5°; 4 P.M., 2°; 6 P.M., $-1°$; 8 P.M., $-3°$; 10 P.M., $-7°$; midnight, $-10°$; 2 A.M., $-5°$; 4 A.M., $-2°$; 6 A.M., 1°; 8 A.M., 7°; 10 A.M., 14°; noon, 22°.

56. Other Uses of Negative Numbers. If we call some special point on a line zero (0), we usually call distances measured to the right positive and distances measured to the left negative, just as we call distances up (as on the thermometer) positive and distances down negative. Again, because we usually call weight positive, we speak of the weight of a balloon (which pulls upward) negative.



The following are some additional illustrations of negative numbers:

If a man is worth \$1000, we may say that he has + \$1000; but if he is \$1000 in debt, we may say that he is worth − \$1000.

If we have 25 on a score in a game, we have + 25; but if we are 25 worse off than nothing, we have − 25.

If we call latitude north of the equator positive, we should call south latitude negative.

If we call longitude west of Greenwich positive, we should call east longitude negative; and if we call west longitude negative, we should call east longitude positive.

If we call the motion of a piston rod of an engine positive when it is to the right, we should call it negative when it is to the left.

If we call downward pressure positive, we should speak of upward pressure as negative.

If we call distance above the earth's surface positive, we should call distance below the earth's surface negative.

We therefore see that negative numbers are just as real as positive numbers, for a man's debts are just as real as his capital, and the temperature is just as real when the thermometer indicates that the temperature is below zero as it is when the mercury rises above zero.

In ancient times people used only whole numbers (integers). Other kinds of numbers, like $\frac{3}{4}$, $\sqrt{3}$, and -3, were invented as they became necessary, and such numbers are sometimes called *artificial numbers*.

57. Adding Negative Numbers. If we tie to a 10-ounce weight a toy balloon that pulls upward 1 oz., what will the weight and balloon together weigh? From the answer to this question we have the following:

To add a positive number to a negative number, find the difference between their absolute values and prefix the sign of the numerically greater number.

Thus $+6$ oz. and -1 oz. are $+5$ oz.;
$+6$ oz. and -6 oz. are 0 oz.;
$+6$ oz. and -9 oz. are -3 oz.

Similarly, *to add a negative number to a negative number, find the sum of their absolute values and prefix the negative sign.*

As a good illustration of this principle, use two balloons, one pulling upward 5 oz. and the other pulling upward 6 oz.

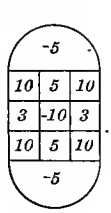
Exercise 37. Addition

Examples 1 to 5, oral

1. What is the combined weight of $+25$ lb. and -5 lb.?

2. What is the combined weight of 30 lb. and -60 lb.?

3. A man who was \$450 in debt contracted another debt of \$250 and then earned \$1000. How much was he then worth?



4. A game is played by throwing bean bags in the direction of the arrow. Suppose that the score stands -5, 5, 3, 10, -10, 5, 10, 10, 3, 3, -5, how much is the total score?

5. If this board without any weights at the ends just balances, and if I put 5 lb. at one end and 8 lb. at the other end, how much must I add to the 5 lb. to make it balance? Instead of adding to the 5 lb., how much must I add to the 8 lb.?



6. A boat that can make 14.1 mi. an hour in still water is going against a stream that retards it 1.8 mi. an hour. What is the rate at which the boat will travel?

14.1 mi. and -1.8 mi. are how many miles?

7. Find the average noon temperature for the week in which the noon temperatures were 15°, 3°, 0°, $-7°$, $-20°$, 6°, 25°.

To find the average of seven numbers, divide their sum by 7, and similarly to find the average of any other set of numbers.

8. A mass of iron and wood is placed in a tank of water. The iron tends to sink the mass with a force of 20 lb., and the wood tends to buoy it up with a force of 16 lb. What does the mass weigh under water?

9. If my watch is 5 min. faster than the schoolroom clock, and the clock is 7 min. slower than the correct time, how near is my watch to the correct time?

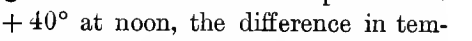
10. In a tug of war one group of boys pulls to the north with a force of $256\frac{3}{4}$ lb., and the other group to the south with a force of $252\frac{1}{2}$ lb. What is the resulting force?

11. An airplane that can fly 58.2 mi. an hour in still air is flying against a wind that retards it 9.7 mi. an hour. At what rate does the airplane fly?

12. An office building contains 37 stories above the street and 4 stories below. An elevator starting at the street level ascends 28 stories, descends 30, ascends 35, descends 37, and ascends 26. Where is the elevator then? Represent the movements by a diagram, using positive and negative signs.

13. A man having \$374.75 in the bank deposits \$176.50 on Monday, checks out \$482.60 on Tuesday, deposits \$243.85 on Wednesday, and checks out \$281.45 on Thursday. Add the positive numbers, then the negative numbers, and then the two sums, and thus find his balance.

58. Subtracting a Negative Number. If the temperature is $-10°$ at midnight and $+40°$ at noon, the difference in temperature is evidently 50°, for the mercury must rise 10° to reach 0°, and 40° more to reach $+40°$. Likewise, in this figure the difference between -2 and $+4$ is 6; for a point must move 2 spaces to get from -2 to 0, and 4 more to reach $+4$; that is, 6 must be added to -2 to make 4.



To subtract a negative number we may add the positive number which has the same absolute value.

That is, $4-(-2)=4+2=6$. Likewise $-5-(-3)=-5+3=-2$. This is the usual rule for subtracting a negative number, but it is never necessary actually to change the sign of the subtrahend.

Exercise 38. Subtraction

Examples 1 to 10, oral

1. How much difference in price is there in selling a horse at \$25 below cost or at \$30 above cost?

2. One morning the temperature was $+11°$, and the next morning $-7°$. What was the difference in temperature?

3. If there is a house for every number, how many houses will you pass in going from 48 East Washington Street to 17 West Washington Street, including both these houses?

4. Jefferson Street is 6 blocks east of Adams Street, and Monroe Street is 9 blocks west of Adams Street. How many blocks west of Jefferson Street is Monroe Street?

Find the value of each of the following:

5. $4-(-3)$.
6. $5-(-7)$.
7. $6-(-6)$.
8. $6-(-3)$.
9. $7-(-7)$.
10. $8-(-9)$.
11. $13.7-(-27.8)$.
12. $16.5-(-43.4)$.
13. $43.4-(-16.5)$.

59. Subtracting a Positive Number. If the temperature is $-10°$ at midnight and falls $2°$ during the next hour, it is then $-12°$. That is, to subtract $2°$ from $-10°$ is the same as to add $-2°$ to $-10°$. Therefore

To subtract a positive number we may add the negative number which has the same absolute value.

Hence $(+a)-(+b)=(+a)+(-b)=a-b$. It is not necessary to remember such a rule, for if we are taking a smaller number from a larger, the result is positive, and in the opposite case it is negative.

Exercise 39. Subtraction

Examples 1 to 13, oral

1. How much is $10°-5°$? $0°-5°$? $-5°-5°$?
2. How much is \$20 – \$10? \$0 – \$10? \$5 – \$10?
3. How much is 9 ft. – 6 ft.? 0 ft. – 6 ft.? – 9 ft. – 6 ft.?
4. How much is $12¢-7¢$? $7¢-7¢$? $5¢-7¢$?
5. How much is \$25 – \$15? \$15 – \$15? \$5 – \$15?

State the value of each of the following:

6. $17-7$.
7. $17-17$.
8. $6-5$.
9. $4-5$.
10. $7-3$.
11. $1-3$.
12. $-4-7$.
13. $-5-6$.

14. If a man is worth \$178 and incurs a debt of \$275, how much is he then worth?

15. If a man is worth – \$178 (is \$178 in debt) and incurs a debt of \$275, how much is he then worth?

Find the value of each of the following:

16. $69-32$.
17. $69-72$.
18. $48-75$.

19. If two trains meet at 9 45 A.M., one running 47.6 mi. an hour and the other 39.7 mi. an hour, how far apart will they be at 10 15 A.M. if these rates are maintained?

60. Multiplying and Dividing Negative Numbers. We multiply and divide negative numbers just as we multiply and divide positive numbers. If a man has $-\$5$ (is \$5 in debt), he will have $-\$10$ if he is twice as much in debt.

That is, $2 \times (-\$5) = -\$10,$

and $-\$10 \div 2 = -\$5.$

Exercise 40. Multiplication and Division

Examples 1 to 4, oral

1. If one balloon pulls upward with a force of 400 lb. (weighs -400 lb.), what will be the upward pull of 3 such balloons? Represent the weight as a negative number.

2. If the thermometer indicates $-8°$, what is the temperature when it indicates half as much below zero? when it indicates twice as much below zero?

3. If a carrier pigeon can fly 78 mi. an hour in still air, at what rate will it fly against a wind that retards it 15 mi. an hour? against a wind that retards twice as much?

4. A checker of bales of cotton finds one bale 16 lb. short, a second bale twice as much short, and a third bale half as much short in weight. Express these shortages of weight in algebraic language.

5. Last year a man's debts amounted to \$475. The year before they were 4 times as much. This year he has paid his debts and has \$825 in the bank. What is the difference between his financial standing year before last and now?

Find the value of each of the following:

6. $3 \times (-86)$.
7. $4 \times (-49)$.
8. $6 \times (-73)$.
9. $7 \times (-89)$.
10. $8 \times (-96)$.
11. $1.7 \times (-3.2)$.
12. $2.8 \times (-4.9)$.
13. $-125 \div 25$.
14. $-493 \div 17$.

61. Multiplying by a Negative Number. We cannot pick up a book -2 times, but we may define what we mean by multiplying by -2, and then we may use the word "times" with negative numbers as we do with positive integers.

Because $3 \times (-2) = -6$, it should follow that $-2 \times 3 = -6$. Therefore we define multiplication by a negative number to mean multiplication by the positive number having the same absolute value, the sign of the product being then changed.

That is, $-2 \times 3 = -6$, and $-2 \times (-3) = 6$.

We know that $2 \times 3 = 6$, and $2 \times (-3) = -6$. Therefore

$$a \cdot b = ab, \quad a \cdot (-b) = -ab.$$

If two numbers have like signs, their product is positive; if they have unlike signs, their product is negative.

Exercise 41. Multiplication

Examples 1 to 4, oral

1. If each of 3 men spends \$2 in a store, how much does the store receive? How much is $3 \times \$2$?

2. If each of 3 men steals \$2 from a store, how much does the store gain or lose? How much is $3 \times (-\$2)$?

3. If 3 men who would have spent \$2 each in a store are persuaded to trade elsewhere, how much does the store gain or lose in gross receipts? How much is $-3 \times \$2$?

4. If 3 men who would have stolen \$2 each from a store are arrested before they can make the theft, how much does the store gain or lose? How much is $-3 \times (-\$2)$?

Find the results of the following:

5. $3 \cdot (-9)$.
6. $-3 \cdot 9$.
7. $-3 \cdot (-9)$.
8. $-27 \cdot 63$.
9. $-42 \cdot (-76)$.
10. $-29 \cdot (-38)$.

62. Dividing by a Negative Number. Division being the inverse of multiplication, we have the following:

$$2 \times 3 = 6, \qquad 6 \div 3 = 2;$$
$$2 \times (-3) = -6, \qquad -6 \div (-3) = 2;$$
$$-2 \times 3 = -6, \qquad -6 \div 3 = -2;$$
$$-2 \times (-3) = 6, \qquad 6 \div (-3) = -2.$$

If two numbers have like signs, their quotient is positive; if they have unlike signs, their quotient is negative.

Exercise 42. Division

Examples 1 to 16, oral

1. How much is $2 \cdot (-7)$? $-14 \div 2$? $-14 \div (-7)$?
2. How much is $-3 \cdot 8$? $-24 \div (-3)$? $-24 \div 8$?
3. How much is $-7 \cdot (-9)$? $63 \div (-7)$? $63 \div (-9)$?
4. How much is $-a \cdot (-b)$? $ab \div (-a)$? $ab \div (-b)$?

State the results of the following:

5. $25 \div 5$.
6. $25 \div (-5)$.
7. $-25 \div 5$.
8. $-25 \div (-5)$.
9. $36 \div (-4)$.
10. $-36 \div 4$.
11. $-36 \div (-4)$.
12. $-36 \div (-9)$.
13. $56 \div (-8)$.
14. $56 \div (-7)$.
15. $-56 \div (-8)$.
16. $-56 \div (-7)$.
17. By what must -22 be multiplied to make 770? -770?

Find the results of the following:

18. $625 \div 25$.
19. $625 \div (-25)$.
20. $-625 \div 25$.
21. $3367 \div 37$.
22. $3367 \div (-37)$.
23. $-3367 \div (-37)$.
24. $-3367 \div (-91)$.
25. $34.3 \div (-7)$.
26. $-34.3 \div (-7)$.
27. $-34.3 \div (-0.7)$.

63. System of Integers. We now see that our system of integers extends indefinitely on both sides of zero, thus:

$$\cdots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \cdots$$

64. Properties of Zero. The following are the important properties of zero:

$$a + 0 = a, \qquad 0 - a = -a, \qquad 0 \cdot a = 0,$$
$$a - 0 = a, \qquad a \cdot 0 = 0, \qquad 0 \div a = 0.$$

Since division by zero is not allowed, $a \div 0$ has no meaning.

Exercise 43. Review

Examples 1 to 17, oral

State the results of the following:

1. $4 + (-3)$.	**6.** $-5 + 7 \times 3$.	**11.** -8×9.
2. $6 + (-9)$.	**7.** $-5 - 9 \div 9$.	**12.** $-7 \times (-8)$.
3. $9 + (-3)$.	**8.** $-7 - 2 \times 4$.	**13.** $8 \div (-2)$.
4. $1 + (-9)$.	**9.** $2 \times (-7) \times 3$.	**14.** $-8 \div 2$.
5. $7 + (-3)$.	**10.** $5 \times (-3) - 2$.	**15.** $-6 \div 2$.

16. How much is $-3 \times (-3)$? $(-3)^2$? $(-7)^2$? $(-9)^2$?

17. How much is $-5 \times 7 - 4$? $-8 + 4 \div 2 \times 3$?

Find the results of the following:

18. $(-1)^2$.	**20.** $(-2)^4$.	**22.** $(-5)^2$.	**24.** $(-1)^6$.
19. $(-1)^5$.	**21.** $(-2)^5$.	**23.** $(-6)^2$.	**25.** $(-1)^7$.

26. If $x = -2$, what is the value of x^2? of $-x$? of $(-x)^2$?

If $a = 4$ and $b = -3$, find the values of:

27. $a + b$.	**29.** $-b$.	**31.** $(-b)^2$.	**33.** $-3ab$.
28. $a^2 - 2b$.	**30.** $a^2 - \frac{1}{3}b$.	**32.** $3a^2b^2$.	**34.** $a^2 - b^2$.

Exercise 44. Review of Chapter IV

1. Draw a straight line to represent the equator of the earth as shown on a map, and draw two parallel lines above it and two below it, at equal distances, to represent degrees of north and south latitude. Trace on the drawing the course of a ship which starts at the equator, sails west, and is blown by a storm, successively 1° N., 2° S., $\frac{1}{2}$° N., 1° S.

2. Representing gains as positive numbers and losses as negative numbers, find the net change in the price of stock whose changes in value for a month were $+3$, -2, $+5$, $-1\frac{1}{2}$, $+5$, -3, -2, $+\frac{1}{8}$.

3. A cash-register attendant whose wages were \$3.50 per day found that she lacked \$5 of the cash called for by the sales slips in her possession. This amount was deducted from her wages. What was the effect on her week's wages of this day's operations? Which of these two amounts may be called positive and which negative?

If $a = 5$ and $b = -3$, find the values of:

4. $(a+b)(a-b)$.

5. $a^2 + 2ab + b^2$.

6. $a^3 + 3a^2 + b$.

7. $a^3 + 3a^2b + 3ab^2 + b^3$.

8. In a battle lasting 5 da. an army division numbering 20,000 men was engaged and was affected as follows:

1st day, loss 2000, reënforcements 1000;
2d day, loss 2500, reënforcements 3000;
3d day, loss 3400, detached by commander 5000;
4th day, loss 2000, reënforcements 7000;
5th day, loss 2200.

Represent these numbers algebraically, and find the effect on the strength of the division for each day and the total effect of the actions of the 5 da.

Exercise 45. Review of Chapters I-IV

1. In the formula $i = prt$ it is given that $i = \$80$, $p = \$700$, $r = 0.04$. Substitute these values and find the value of t.

2. Using the formula of Ex. 1, if $i = \$90$, $r = 0.05$, and $t = 2\frac{1}{4}$ yr., find the value of p.

3. Write the formulas for finding the area of a rectangle, the area of a triangle, the area of a trapezoid, and the area of a circle. Make problems which can be solved by these formulas and solve them.

Find the value of x in each of the following equations:

4. $x + 3 = 11\frac{1}{2}$.

5. $x - 0.85 = 0.375$.

6. $\frac{x}{3.2} = -9$.

7. $x + 5 = -3$.

8. $7x + 2 = 30$.

9. $5x - 2.7 = 8.3$.

10. $\frac{2x}{1.1} = 2.4$.

11. $\frac{3}{4}x - 2 = \frac{1}{2}x + 6$.

12. By what number must 17 be multiplied to make 544? to make -544? to make -1088?

13. By what number must -19 be multiplied to make -399? to make 798?

14. On every article sold in a stationery store during the month of May there was a gain of 18% on the cost. If the total sales amounted to \$3360, find the cost of the goods sold.

15. If increases of sales for a month, as compared with the previous month, are marked plus and decreases minus, find the sales for December, those of January of the same year being \$47,000 and the changes for the other months being: February, $-$ \$500; March, $-$ \$800; April, $+$ \$2000; May, $-$ \$400; June, $+$ \$1500; July, $-$ \$1000; August, $+$ \$50; September, $+$ \$1100; October, $+$ \$450; November, $+$ \$1200; December, $+$ \$1850.

CHAPTER V

ADDITION AND SUBTRACTION

65. Addition of Similar Monomials. In arithmetic we learned that 5 apples + 3 apples = 8 apples, but that 5 horses + \$3 gives no simple sum expressed in terms of a single unit.

That is, only monomials which are expressed in the same unit can be added so as to give a simple sum.

Monomials which have a common factor are called *similar terms* or *similar monomials* with respect to that factor.

Thus a, $3a$, and $-7a$ are similar with respect to a; $2\sqrt{5}$ and $-\frac{2}{3}\sqrt{5}$ are similar with respect to $\sqrt{5}$; $2\cdot(-7)$ and $4\cdot(-7)$ are similar with respect to -7; and ax^2 and bx^2 are similar with respect to x^2.

Monomials that are not similar are said to be *dissimilar*.

Exercise 46. Addition of Similar Monomials

All work oral

1. Add 2 ft. and 3 ft.; $2f$ and $3f$; $2\cdot 5$ and $3\cdot 5$.
2. Add $7¢$ and $9¢$; $7c$ and $9c$; $7\cdot 2$ and $9\cdot 2$; $7\cdot 10$ and $9\cdot 10$.
3. Add \$15 and \$11; $15d$ and $11d$; $15\cdot 2$ and $11\cdot 2$.
4. How do you proceed to add similar monomials?

Add the following:

5. 7 mi. + 14 mi.
6. $7m + 14m$.
7. $7\cdot 2 + 14\cdot 2$.
8. 6 bu. + 4 bu. + 9 bu.
9. $6b + 4b + 9b$.
10. $6\cdot 7 + 4\cdot 7 + 9\cdot 7$.

66. Algebraic Sum. The result obtained by adding two or more numbers considered with respect to their signs as well as their values is called their *algebraic sum.*

Thus the algebraic sum of 2 and -3 is -1, although $2 + 3 = 5$.

67. Addition of Monomials. *To add similar monomials, find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.*

For this purpose we may consider an expression like $27(a - b)$ as a monomial, as in Ex. 15, below. In case a letter has no coefficient, 1 is understood. Thus $5x + x = 6x$.

To add dissimilar monomials, write the terms one after the other, each with its proper sign.

Thus the sum of a, $2b$, and $-c$ is $a + 2b - c$.

Exercise 47. Addition of Monomials

Examples 1 to 5, oral

1. Add \$2 and \$3; $2d$ and $3d$; $2 \cdot 25$ and $3 \cdot 25$.

2. Add 3 yd., 2 ft., and 7 in., expressing the result as a compound number. Add $3y$, $2f$, and $7i$; $3a$, $2b$, and $7c$.

3. Add a, $-2b$, and $3c$; x, $-2y$, and $3z$; m, $-2n$, and $3p$.

4. Add $-2a$, b, and $4c$; $-2x$, y, and $4z$; $-3x$, y, and z.

5. Add $5a$, $6b$, and $-7a$; $5p$, $6q$, and $-7r$; a, $4a$, and c.

Add the following:

6. a, $-a$, 2, -2.
7. $-a$, a, -2, 2.
8. $19a$, $-3x$, $3x$.
9. $-4a$, $42x$, $4a$.
10. $17a$, $-19x$, $19x$.
11. $18a^2b$, $36a^2b$, $-7a^2b$.
12. $2\sqrt{a}$, $13\sqrt{a}$, $15\sqrt{a}$.
13. $16\sqrt{m}$, $-23\sqrt{m}$, $\sqrt{m}$.
14. $92abc$, $37abc$, $-75abc$.
15. $27(a - b) + 3(a - b)$.

68. Addition continued. In commercial work we frequently find it necessary to make additions like the following:

7×25	$112 \times \frac{1}{365} \times 0.05$	$975 \times 0.04 \times 8$
4×25	$238 \times \frac{1}{365} \times 0.05$	$654 \times 0.04 \times 8$
9×25	$196 \times \frac{1}{365} \times 0.05$	$781 \times 0.04 \times 8$
20×25	$546 \times \frac{1}{365} \times 0.05$	$2410 \times 0.04 \times 8$
$= 500$	$= 7.48\%$	$= 771.2$

In the first case we add exactly as if we had $7x + 4x + 9x$, only that we have 25 instead of x. In the second case we add exactly as if we had $112x + 238x + 196x$, only that we have $\frac{1}{365} \times 0.05$ instead of x, and similarly for the third case. We see how much easier it is to add in this way than to perform all the multiplications and then add.

Exercise 48. Addition of Monomials

Examples 1 to 5, oral

Add the following:

1.	2.	3.	4.	5.
$2ax^2$	$12xyz^2$	$2pqr$	$-2abc$	3×10
$5ax^2$	$12xyz^2$	$13pqr$	$-3abc$	9×10
$7ax^2$	$8xyz^2$	$10pqr$	$-5abc$	3×10
$3ax^2$	$8xyz^2$	$5pqr$	$-15abc$	5×10

Add the following and find the value of each result:

6.	7.	8.
5×73	$198 \times \frac{1}{365} \times 0.06$	$975 \times 0.04 \times 8$
9×73	$256 \times \frac{1}{365} \times 0.06$	$654 \times 0.04 \times 8$
6×73	$458 \times \frac{1}{365} \times 0.06$	$781 \times 0.04 \times 8$
15×73	$954 \times \frac{1}{365} \times 0.06$	$963 \times 0.04 \times 8$
42×73	$845 \times \frac{1}{365} \times 0.06$	$777 \times 0.04 \times 8$
81×73	$976 \times \frac{1}{365} \times 0.06$	$549 \times 0.04 \times 8$
91×73	$721 \times \frac{1}{365} \times 0.06$	$1245 \times 0.04 \times 8$

Exercise 49. Addition of Polynomials

Examples 1 and 2, oral

1. Add 2 ft. 3 in. and 3 in.; $2f+3i$ and $3i$.

2. Add 7 yd. 2 ft. and 6 yd.; $7y+2f$ and $6y$.

Add the following:

3.

$$\begin{array}{l} 6 \text{ ft.} + 7 \text{ in.} \\ \underline{2 \text{ ft.} + 3 \text{ in.}} \end{array}$$

4.

$$\begin{array}{l} 6f+7i \\ \underline{2f+3i} \end{array}$$

5.

$$\begin{array}{l} 6 \cdot 5 + \ 7 \cdot 8 \\ \underline{2 \cdot 5 + \ 3 \cdot 8} \\ (?) \cdot 5 + (?) \cdot 8 \end{array}$$

6.

$$\begin{array}{l} 2 \text{ rd.} + 8 \text{ ft.} + 9 \text{ in.} \\ \underline{4 \text{ rd.} + 3 \text{ ft.} - 2 \text{ in.}} \end{array}$$

7.

$$\begin{array}{l} 2r+8f+9i \\ \underline{4r+3f-2i} \end{array}$$

8.

$$\begin{array}{l} 2 \cdot 3 + 8 \cdot 7 + 9 \cdot 4 \\ \underline{4 \cdot 3 + 3 \cdot 7 - 2 \cdot 4} \end{array}$$

9.

$$\begin{array}{l} x^2+10xy+15z^2 \\ \underline{x^2+\ 4xy-10z^2} \end{array}$$

10.

$$\begin{array}{l} 3w+2x+3y+4z \\ \underline{2w-3x+2y+3z} \end{array}$$

11.

$$\begin{array}{l} 7^2+\ 5 \cdot 7 \cdot 3 \\ \underline{7^2-\ 2 \cdot 7 \cdot 3} \\ (?) \cdot 7^2 + (?) \cdot 7 \cdot 3 \end{array}$$

12.

$$\begin{array}{l} x^2+10y+7z+9 \\ \underline{x^2-\ 3y+4z-9} \end{array}$$

13.

$$\begin{array}{l} x^4+x^2y^2+y^4+x^2+y^2 \\ \underline{x^4-x^2y^2+y^4-x^2-y^2} \end{array}$$

14.

$$\begin{array}{l} a+3b-c+d-e+f \\ \underline{a-3b+c-d+e-f} \end{array}$$

15. If $A=x^3+2x^2-7x+1$ and $B=2x^3+x^2+10x+2$, what is the value of $A+B$?

69. Addition of Polynomials. In adding polynomials it is usually desirable to proceed as follows:

Write similar terms in the same column and add these terms, writing their sums as a polynomial.

70. Check. An operation that tends to prove the correctness of another operation is called a *check* upon that operation.

71. Checks in Algebra. One of the best checks is the substitution of any values we please for the letters, as in the following example, where we let $x = 1$, $y = 1$, and $z = 1$.

Operation	Check
$2x + 3y - 4z + 6$	$2 + 3 - 4 + 6 = 7$
$3x - 7y + 8z - 3$	$3 - 7 + 8 - 3 = 1$
$5x - 4y + 4z + 3$	$5 - 4 + 4 + 3 = 8$

Here we have simply put 1 in place of x, y, and z in the addends and in the sum, and we have $7 + 1 = 8$. Therefore the work *checks*.

We may have an error in spite of this check, as would be the case if we should write $5y - 4x + 4z + 3$ instead of $5x - 4y + 4z + 3$, or if we should make an error in computation in the check. In case of doubt, especially in case of exponents, use other values than 1.

Exercise 50. Addition of Polynomials

Examples 1 to 3, oral

1. Add $2x + y$ and $3x + y$; $2x + y$ and $3x - y$.
2. Add $7x + 3y$ and $3x + 7y$; $7x + 3y$ and $-3x - 7y$.
3. Add $a + b + c$ and $a - b + c$; $a - b - c$ and $a + b + c$.

Add the following and check the results:

4. $x^2 + xy + y^2$, $x^2 - 2xy + y^2$, $x^2 + 2xy + y^2$.
5. $x^2 + y^2$, $x^2 - y^2$, $2x^2 + y^2$, $x^2 + 2y^2$, x^2, $3y^2$.
6. $x^3 + x^2y + xy^2 + y^3$, $2x^3 - 3x^2y + 7xy^2 - 6y^3$.

Add the following and check the results:

7. $5x^2+3x+7,\ 6x^2-3x+7,\ x+2.$

8. $7x^2-9x+2,\ 6x^2-8x+3,\ x-9.$

9. $2x^2+3x-9,\ 4x^2-6x-8,\ x+4.$

10. $3x^2+5x+6,\ 7x^2-7x+9,\ x-4.$

11. $4x^2-9x-8,\ 2x^2-x+1,\ x^2+3x-1.$

12. $-x^2+2x+7,\ -x^2-x+9,\ 3x^2+2x+3.$

13. $-5x^2-7x+9,\ -10x^2-14x+13,\ 2x^2-8.$

14. $a^2+ab+b^2,\ a^2-ab+b^2,\ -a^2+ab-b^2,\ a^2+b^2.$

15. $a^3+a^2+a+1,\ a^3-a^2+a-1,\ a^3+a^2-a+1,\ -3a^3.$

16. $p^2+q^2+4,\ p^2-q^2+7,\ -p^2+q^2-8,\ -p^2-q^2-3.$

17. $m^2n+mn^2+7,\ m^2n+mn^2-7,\ m^2n-2mn^2,\ -3m^2n.$

Simplify the following by combining similar terms:

18. $a^2+3b^2-4c^2+2d^2-7+3a^2-4b^2+c^2-2d^2+7.$

19. $a^3+3a^2b+3ab^2+b^3-a^3+3a^2b-3ab^2+b^3.$

20. $x^3+2x^2+3x-1+4x+3x^2+1-x^3-5x^2-7x.$

21. $m^2n+mn^2+mn-3mn^2-mn+4m^2n.$

22. Add $3t^3+5t^2+4t+2$ and $2t^3+3t^2+2t+5$; also 3542 and 2325. Compare the figures of the two results.

23. How much is 2 ft. + 3 ft.? a feet + b feet? $2f+3f$? $af+bf$? $ax+bx$? $am+bm$? $2\cdot5+3\cdot5$? $a\cdot5+b\cdot5$?

We may think of a feet + b feet as $(a+b)$ feet. Similarly, we have $af+bf=(a+b)f$. That is, we do not know the numerical value of the coefficients a and b, so we *indicate* their sum.

Add the following, indicating the sums of the coefficients:

24. p yards + q yards; $py+qy$; $pm+qm$; $p\cdot5+q\cdot5$.

25. $ax+bx+cx$; $am^2+bm^2+cm^2$; $a\sqrt{m}+b\sqrt{m}+c\sqrt{m}$.

72. Relation of Polynomials to Numbers. If we use u for units, t for tens, and h for hundreds, we can write eight hundred forty-seven in these three ways: 847, $800 + 40 + 7$, and $8h + 4t + 7u$. But since our numbers are arranged on a scale of 10, so that 10 units of one order make 1 of the next order, we may also write this $8t^2 + 4t + 7$, where $t = 10$.

We also see this relation in the addition of numbers, thus:

845	$8h + 4t + 5u$	$8t^2 + 4t + 5$
943	$9h + 4t + 3u$	$9t^2 + 4t + 3$
560	$5h + 6t + 0u$	$5t^2 + 6t + 0$
2348	$22h + 14t + 8u$	$22t^2 + 14t + 8$

The addition is the same in all three forms, but in the case of the numbers we remember that $10t = 1h$.

The teacher should show that the polynomial form allows for an excellent explanation of the carrying process in addition. It also shows that in simple cases in addition it is often easier to begin at the left, looking ahead to see what must be allowed for the carrying.

Exercise 51. Relation to Numbers

Express the following numbers as polynomials, using h, t, and u, as above, and add:

1.	2.	3.	4.	5.
729	140	300	604	878
632	270	270	327	569
421	620	425	426	390

Express the following numbers as polynomials, using t^3, t^2, and t, and add:

6. $2732 + 4876 + 2984$.
7. $1286 + 2707 + 3041$.
8. $2648 + 3000 + 2009$.
9. $3478 + 2000 + 5270$.
10. $4168 + 5096 + 3087$.
11. $5275 + 289 + 46 + 3$.

73. Check in Addition. By considering the example in § 72 we see that it is immaterial whether we start to add at the right or at the left, provided the term resulting from each column is put in its proper place in the number system.

This fact is practically used in commercial work as a check on addition, as follows:

	Checks			Checks	
9437			4507		
285	16	23	78	37	4
7381	29	23	356	21	9
940	23	29	129	9	21
7563	23	16	77	4	37
25606	25606	25606	5147	5147	5147

In each of these examples in addition the left-hand check is the one usually employed. In the first example it may be expressed algebraically as $16\,u + 29\,t + 23\,h + 23\,T$.

Exercise 52. Checks

Add the following and check, arranging the work as shown above and expressing each check as a polynomial:

1.	2.	3.	4.	5.
29,842	59,478	75,648	597,378	78,568
3,785	5,765	587	5,096	457
6,597	946	9,467	72,194	9,723
508	88	92	879	976

6.	7.	8.	9.	10.
42,749	12,926	57,725	$423.76	$363.48
68,923	96,009	38,936	28.94	249.76
40,007	3,470	40,037	362.05	82.93
3,428	62,893	629	436.75	834.39
60,000	48,762	43,334	82.96	90.00

74. Addition of Denominate Numbers. Since the scale of relation in denominate numbers is frequently inconvenient, not being 10 as in our common system, the work of addition can frequently be shortened by the use of negative numbers as shown below.

Arithmetic	Better Form	Gal. Qt.
24 gal. + 3 qt.	25 gal. − 1 qt.	25 − 1
42 gal. + 1 qt.	42 gal. + 1 qt.	42 + 1
51 gal. + 2 qt.	51 gal. + 2 qt.	51 + 2
30 gal. + 3 qt.	31 gal. − 1 qt.	31 − 1
17 gal. + 3 qt.	18 gal. − 1 qt.	18 − 1
167 gal. + 0 qt.	167 gal. + 0 qt.	167

If necessary, the teacher should explain, without formal definition, the meaning of "scale of relation."

Exercise 53. Denominate Numbers

Using the third of the above methods, add the following:

1.	2.	3.
35 gal. 3 qt.	5 da. 4 hr.	£180 19 s.
18 gal. 2 qt.	3 da. 20 hr.	£485 7 s.
42 gal. 1 qt.	6 da. 3 hr.	£450 18 s.
178 gal. 3 qt.	5 da. 19 hr.	£248 16 s.

4.	5.	6.
2 A. 15 sq. rd.	9 T. 15 cwt.	3 pk. 6 qt.
1 A. 150 sq. rd.	6 T. 11 cwt.	2 pk. 2 qt.
9 A. 155 sq. rd.	5 T. 3 cwt.	5 pk. 7 qt.
7 A. 26 sq. rd.	4 T. 18 cwt.	1 pk. 2 qt.
6 A. 7 sq. rd.	7 T. 2 cwt.	5 pk. 5 qt.

7. 65° 21′, 45° 39′, 18° 10′, 5° 50′, 11° 38′, 15° 20′, 9° 40′.

75. Interest on Daily Balances. Many banks of deposit allow interest on daily balances provided the average daily balance for the month reaches a certain specified amount. The depositor is then allowed one day's interest on the balance for each day, the rate usually being 2% per annum and reckoned on a 365-day basis.

In paying interest, banks commonly reckon 365 da. to a year, using tables arranged on that basis, but in charging interest on loans they use tables based on 360 da., the latter giving the greater interest.

Thus, if the balances for the first 5 da. of the month are as follows, the interest allowances would be as shown below:

Day	Balance	Interest reckoned	Algebraically
1	\$48,000	$48{,}000 \times 0.02 \times \frac{1}{365}$	$48{,}000\,rt$
2	57,000	$57{,}000 \times 0.02 \times \frac{1}{365}$	$57{,}000\,rt$
3	89,000	$89{,}000 \times 0.02 \times \frac{1}{365}$	$89{,}000\,rt$
4	125,000	$125{,}000 \times 0.02 \times \frac{1}{365}$	$125{,}000\,rt$
5	75,000	$75{,}000 \times 0.02 \times \frac{1}{365}$	$75{,}000\,rt$
Interest allowance,		$394{,}000 \times 0.02 \times \frac{1}{365}$	$394{,}000\,rt$

It is evident from the above work that the bookkeeper who has to compute the interest due the depositor at the end of the month need not go to all the trouble of computing the interest for each day and finding the sum of these interest amounts. Instead of going through this laborious process he can find the result by adding the daily balances and getting one day's interest on the sum. In the above example the interest for 5 da. is \$21.58, the bank taking the fraction of 1¢.

Herein we see one of the many practical uses of algebra, the showing of the reason why it is legitimate to take this short method. As we proceed we shall find that algebra is continually coming to the aid of those who have to make commercial calculations, revealing short methods and giving the power of discovering such methods, as new conditions arise to require them.

Exercise 54. Interest on Daily Balances

By the algebraic method given on page 80, compute the interest due on the daily balances for the month as shown below:

ACCOUNT OF *Henry Student* WITH SCHOOL BANK, *July*, 19*9*

Interest not allowed unless average daily balance is $20,000 or more.

DATE	THOUSANDS	HUNDREDS	DATE	THOUSANDS	HUNDREDS
1	158	1	*Brought up*	*	*
2	92	6	17	57	3
3	52	3	18	142	7
4	52	3	19	76	7
5	75	4	20	76	7
6	75	4	21	155	0
7	32	5	22	38	9
8	7	5	23	256	7
9	68	4	24	78	4
10	92	7	25	148	5
11	75	6	26	235	1
12	148	0	27	235	1
13	148	0	28	73	2
14	84	9	29	196	4
15	14	7	30	78	4
16	88	2	31	35	3
Forward	*	*	Total	*	*
We credit interest at 2%, $*.					

Fill the spaces marked by asterisks (*).

Such an account at a bank would appear in one column instead of in two columns as above, but the principle of finding the interest on daily balances would be the same.

76. Subtraction of Similar Monomials. We learned in arithmetic that 8 apples $-$ 3 apples $=$ 5 apples, but that 8 horses $-$ \$3 gives no simple result expressed in terms of a single unit.

That is, in subtraction the minuend and subtrahend must be alike in kind.

Exercise 55. Subtraction of Similar Monomials

Examples 1 to 16, oral

1. How much is 7 ft. $-$ 4 ft.? $7f-4f$? $7\cdot 2-4\cdot 2$?

2. How much is 10 mi. $-$ 3 mi.? $10\,m-3\,m$? $10\,a-3\,a$?

3. How much is 9 yd. $-$ 5 yd.? $9\,y-5\,y$? $9\cdot 3-5\cdot 3$?

State the results of the following:

4. \$30 $-$ \$12.

5. $30\,d-12\,d$.

6. $30\,a-12\,a$.

7. $15\,x-7\,x$.

8. $15\,ax-7\,ax$.

9. $15\sqrt{a}-7\sqrt{a}$.

10. $29\,a-11\,a$.

11. $29\,x^2-11\,x^2$.

12. $29\sqrt{xy}-11\sqrt{xy}$.

13. How do you subtract one monomial from a similar monomial?

14. How much is a feet $-$ b feet? $ax-bx$?

15. From 7° subtract 6°; from 7° subtract 7°; from 7° subtract 8°. What must be added to 6 to make 7? to 7 to make 7? to 8 to make 7?

16. How many degrees from $-1°$ to $+7°$? How much is $7°-(-1°)$? $7-(-1)$? $7\,a-(-a)$?

Find the results of the following:

17. $4.7\,a-3.9\,a$.

18. $5.1\,x^2-2.8\,x^2$.

19. $7.2\,cd-3.9\,cd$.

20. $3\,x-2\frac{4}{5}\,x$.

21. $13\frac{3}{4}\,x-5\frac{1}{2}\,x$.

22. $12\frac{1}{2}\,y-7\frac{7}{8}\,y$.

23. $7°-(-3°)$.

24. $7\,d-(-3\,d)$.

25. $7\,x-(-3\,x)$.

77. Subtraction of Monomials. It is evident from the solutions of the examples in the preceding exercise that

To subtract a monomial from a similar monomial, we find the difference between the coefficients of the common factor and multiply this difference by the common factor.

Thus $9a - 4a = 5a$, $ax - bx = (a - b)x$, $mx^2 - 4nx^2 = (m - 4n)x^2$. In case a letter has no coefficient expressed, 1 is to be understood as usual. Thus $5x - x = 4x$.

If the monomials are dissimilar, we indicate the subtraction.

Thus, as we may write 7 ft. − 3 in., so we write $7f - 3i$, or $7x - 3y$, or $7\sqrt{a+b} - 3\sqrt{x+y}$.

Exercise 56. Subtraction of Similar Monomials

Examples 1 to 3, oral

1. What must be added to 3 to make 7? to $3a$ to make $7a$? to $3a^2b$ to make $7a^2b$? How much is $7a^2b - 3a^2b$?

2. What must be added to $-3°$ to make 7°? to $-3x$ to make $7x$? to $-3\sqrt{a}$ to make $7\sqrt{a}$? How much is $7a - (-3a)$?

3. What must be added to $-10°$ to make 4°? to $-10x$ to make $4x$? How much is $4x - (-10x)$?

Find the results of the following:

4. $30\sqrt{a} - 17\sqrt{a}$.

5. $5.1x^2 - 3.9x^2$.

6. $7\frac{1}{2}m - 3\frac{2}{3}m$.

7. $7\frac{1}{3}m - 5\frac{1}{2}m$.

8. $8\frac{2}{3}q - 7\frac{3}{4}q$.

9. $9a - 9\frac{3}{4}a$.

10. $7.1ab - 3.8ab$.

11. $9.3a^2x - 5.7a^2x$.

12. $a^2b^2 - ya^2b^2$.

13. From the sum of $4a$ and $7a$ subtract $-5a$.

14. From the sum of $7ab$ and $8ab$ subtract the sum of $15ab$, $-2ab$, and $-9ab$.

15. From the sum of $6x^2$, $9x^2$, and $-3x^2$ subtract the sum of $17x^2$, $-6x^2$, $-9x^2$, and $3x^2$.

78. Subtraction as the Inverse of Addition. We may think of subtraction as the operation by which we find the number which, added to the subtrahend, will equal the minuend.

Thus, because 3 added to 4 makes 7, therefore $7 - 4 = 3$.
Because 7 added to -3 makes 4, therefore $4 - (-3) = 7$.
Because 5° added to $-9°$ makes $-4°$, therefore $-4° - (-9°) = 5°$.

We therefore have the following cases in subtraction:

$$\begin{aligned} 4 + 5 = 9; &\quad \text{therefore} \quad 9 - 5 = 4; \\ 4 + (-5) = -1; &\quad \text{therefore} \quad -1 - (-5) = 4; \\ -4 + 5 = 1; &\quad \text{therefore} \quad 1 - 5 = -4; \\ -4 + (-5) = -9; &\quad \text{therefore} \quad -9 - (-5) = -4. \end{aligned}$$

Therefore, *to subtract one quantity from another, find a quantity which, added to the subtrahend, will equal the minuend.*

If the student will think of this rule when he subtracts, he will rarely be troubled in the matter of signs.

Exercise 57. Subtraction of Monomials

Examples 1 to 14, oral

Find the results of the following:

1. $3 - 4$.	**8.** $-3a - (-a)$.	**15.** $-86c - 17c$.
2. $3 - (-4)$.	**9.** $5ax - 3ax$.	**16.** $-6c - (-7c)$.
3. $-3 - 4$.	**10.** $5a - (-3a)$.	**17.** $-79x - 10x$.
4. $-3 - (-4)$.	**11.** $-5a - 3a$.	**18.** $-9x - (-10x)$.
5. $3a - 4a$.	**12.** $7m^2 - m^2$.	**19.** $37a - 19a$.
6. $3a - (-4a)$.	**13.** $7m^2 - (-m^2)$.	**20.** $37a - (-19a)$.
7. $-3a - 4a$.	**14.** $-7m^2 - m^2$.	**21.** $-37a - 19a$.

22. From the difference between $37.5ax$ and $29.3ax$ subtract the difference between $42.9ax$ and $40.7ax$.

Exercise 58. Subtraction of Polynomials

Examples 1 to 10, oral

1. From 4 ft. 8 in. subtract 2 ft. 3 in.

2. From 8 ft. $-$ 4 in. subtract 2 in.; subtract 6 ft. 2 in.

3. From 2° subtract 2°; from $-$ 2° subtract 2°.

4. If the thermometer registers $-$ 2° and the temperature then falls 4°, what does the thermometer then register?

Perform the following subtractions:

5.	11.	17.
7 lb. 4 oz.	$78\,a + 4\,b$	$17\,x^2y^2 - 15\,xy$
3 lb. 2 oz.	$43\,a + 2\,b$	$12\,x^2y^2 + 14\,xy$
6.	**12.**	**18.**
$7\,x + 4\,y$	$68\,a + 4\,b$	$23\,a^2b - 15\,b^2c$
$3\,x + 2\,y$	$35\,a + 4\,b$	$17\,a^2b + 16\,b^2c$
7.	**13.**	**19.**
$7 \cdot 3 + 4 \cdot 2$	$8 \cdot 9 + 4 \cdot 5$	$23 \cdot 10 - 15 \cdot 3$
$3 \cdot 3 + 2 \cdot 2$	$5 \cdot 9 + 4 \cdot 5$	$17 \cdot 10 + 16 \cdot 3$
8.	**14.**	**20.**
$9\,m + 48\,f$	$15\,m + 7\,n$	$71\,ax + mby$
$4\,m - 20\,f$	$9\,m + 6\,n$	$47\,ax - nby$
9.	**15.**	**21.**
$9\,a^2b + 48\,c$	$15\,p - 7\,q$	$29\,x^2y - axy^2$
$4\,a^2b - 20\,c$	$12\,p - 7\,q$	$30\,x^2y - bxy^2$
10.	**16.**	**22.**
15 ft. $-$ 4 in.	$15\,p + 7\,q$	$xm^2n - 21\,mn^2$
8 ft. + 2 in.	$14\,p + 8\,q$	$ym^2n + 15\,mn^2$

79. Problems in Percentage. Problems in percentage may often be more easily solved by algebra than by arithmetic.

1. If 15% of a number is 9165, what is the number?

SOLUTION BY ARITHMETIC

Since 15% of the number = 9165,
therefore 1% of the number = $\frac{1}{15}$ of 9165, or 611,
and the number = 100 × 611, or 61,100.

SOLUTION BY ALGEBRA

Let x = the number.
Then $0.15\,x = 9165$.
Dividing by 0.15, $x = 61{,}100$.

2. What number increased by $66\frac{2}{3}\%$ of itself equals 275?

SOLUTION BY ARITHMETIC

Since 100% of the number = the number,
therefore $66\frac{2}{3}\%$ of the number = the increase,
and $166\frac{2}{3}\%$, or $\frac{5}{3}$, of the number = 275.
Therefore $\frac{1}{3}$ of the number = $\frac{1}{5}$ of 275, or 55,
and the number = 3 × 55, or 165.

SOLUTION BY ALGEBRA

Let x = the number.
Then $x + 0.66\frac{2}{3}\,x = 275$,
or $1.66\frac{2}{3}\,x = 275$.
Dividing by $1.66\frac{2}{3}$, $x = 165$.

3. After deducting 10% from the marked price of a table, a dealer sold it for \$13.50. What was the marked price?

Let x = the *number* of dollars of the marked price.
Then $x - 0.10\,x = 13.50$,
or $0.90\,x = 13.50$.
Dividing by 0.90, $x = 15$.
Therefore the marked price was \$15.

Exercise 59. Percentage

Examples 1 to 10, oral

1. A number less 10% of itself is what per cent of itself?

2. A number plus 10% of itself is what per cent of itself?

State the per cent of x in each of the following:

3. $x+0.05\,x$. **5.** $x+20\%\,x$. **7.** $x+0.50\,x$. **9.** $x+75\%\,x$.

4. $x-0.05\,x$. **6.** $x-20\%\,x$. **8.** $x-0.50\,x$. **10.** $x-75\%\,x$.

11. What number less 10% of itself is equal to 72?

12. What number less 17% of itself is equal to 166?

13. \$435 is 6% of what sum of money?

14. \$19.75 is 5% of what sum of money?

15. \$38.25 is $4\frac{1}{2}\%$ of what sum of money?

16. What is the number of which 14.4 is $66\frac{2}{3}\%$?

17. What is the sum of which \$41.25 is 15%?

18. What is the sum of which $33\frac{1}{3}\%$ is \$2.25?

19. A certain number increased by $12\frac{1}{2}\%$ of itself is equal to 819. What is the number?

20. A boy now weighs 84 lb., which is 12% more than he weighed a year ago. How much did he weigh a year ago?

21. A dealer made a profit of \$7380 this year in his store. This amount is 18% less than the profit made last year. How much was his profit last year?

22. A dealer was obliged to sell some damaged furniture at 10% less than cost and sold it for \$85.50. How much did it cost? How much did he lose?

23. A farmer sold his milk to a dairy company and received credit for 1045 lb. of butter fat. If his milk tested 3.8% butter fat, how many pounds of milk did he sell?

80. Removal of Parentheses preceded by the Positive Sign. If to \$4 we add \$3 + \$2, we have in all \$4 + \$3 + \$2, or \$9. It is evidently of no consequence whether we add \$2 to the sum of \$4 and \$3 or add \$5 to \$4.

Hence $\$4 + (\$3 + \$2) = \$4 + \$3 + \$2.$

Similarly, $5¢ + (7¢ - 2¢) = 5¢ + 7¢ - 2¢,$

Therefore $a + (b + c) = a + b + c,$

and $a + (b - c) = a + b - c.$

If the parentheses inclosing an expression are preceded by the positive sign, the parentheses may be removed without any change in the signs of the terms.

We may treat in a similar way any other sign of aggregation. Thus, $7 + \overline{3 + 4} = 7 + 3 + 4$, and $12 + [5 - 2] = 12 + 5 - 2$.

Exercise 60. Removal of Parentheses

Examples 1 to 6, oral

1. How much is \$4 plus the sum of \$3 and \$9? How much is the sum of \$4, \$3, and \$9?

Remove the parentheses and simplify:

2. $7 + (9 + 6)$.

3. $8 + (15 - 5)$.

4. $4a + (3a + a)$.

5. $6x^2y^2 + (9x^2y^2 - x^2y^2)$.

6. How much is the sum of 7 and 3 increased by the sum of 4 and 2? How much is the sum of 7, 3, 4, and 2?

Remove the parentheses and simplify:

7. $(2a + 3a) + (7a + a)$.

8. $(2a + 3a) + (7a - a)$.

9. $(2a - 3a) + (7a + a)$.

10. $(2a - 3a) + (7a - a)$.

11. $a - b + (a + b)$.

12. $a^2 - b^2 + (a^2 + b^2)$.

13. $5a + 17 + (5a - 17)$.

14. $(6x + 9y) + (5x - 3y)$.

81. Removal of Parentheses preceded by the Negative Sign. If we subtract $b+c$ from a and subtract $b-c$ from a, we have the following results:

$$\begin{array}{r} a \\ b+c \\ \hline a-b-c \end{array} \qquad \begin{array}{r} a \\ b-c \\ \hline a-b+c \end{array}$$

Therefore $$a-(b+c)=a-b-c,$$

and $$a-(b-c)=a-b+c.$$

If the parentheses inclosing an expression are preceded by the negative sign, the parentheses may be removed, provided the sign before each term is changed.

That is, $$x+y-(x-y)=x+y-x+y=2y;$$

and $$\frac{3}{2}-\frac{5-4}{2}=\frac{3-5+4}{2}=\frac{2}{2}=1,$$

the fraction bar having the force of parentheses.

If $$A=x^2+2x-1, \text{ and } B=x^2-7x-8,$$

then $$A-B=x^2+2x-1-x^2+7x+8=9x+7,$$

and $$B-A=x^2-7x-8-x^2-2x+1=-9x-7.$$

Exercise 61. Removal of Parentheses

Examples 1 to 8, oral

1. Subtract $7-5$ from 10. Subtract 7 from 10 and add 5. How do the two answers compare?

2. Subtract $7a-5a$ from $10a$. How much is $10a-7a+5a$? How do the two answers compare?

Remove the parentheses and simplify:

3. $20-(8+2)$.

4. $20-(8-2)$.

5. $30-(9-4)$.

6. $4a-(3a-a)$.

7. $7x-(4x-2x)$.

8. $9x-(5x-3x)$.

9. If $A = x^2 - 3x - 15$, and $B = x^2 - 9x - 11$, what is the value of $A - B$? of $B - A$?

10. In Ex. 9 what is the value of $x^2 - A$? of $x^2 + 7 - A$?

11. In Ex. 9 what is the value of $A - (x^2 - 5x + 7) - B$?

12. In Ex. 9 what is the value of $B - (x^2 + 9x - 6) - A$?

13. Simplify $(17a + 15b) - (12a - b)$.

Remove the parentheses and simplify:

14. $12x - (9x + 7x)$.

15. $17y - (12y - 4y)$.

16. $39a - (15a + b)$.

17. $47a - (15b + 7a)$.

18. $56n + (27n - 4n)$.

19. $39ab + (ab + 1)$.

20. $56a^2 - (3a^2 + 7)$.

21. $48p + (27p - 2)$.

22. $63q - (49 + 4q)$.

23. $79x - (79 - 79x)$.

24. Simplify $25a^2 - \overline{2a^2 + 1}$; $25a^2 + \overline{2a^2 + 1}$.

Remove the bars and simplify:

25. $4a - \overline{3a - a}$.

26. $7a - \overline{4a + 2a}$.

27. $8a + \overline{6a - 3a}$.

28. $9x - \overline{7x - 2x}$.

29. $ab + \overline{3 - ab}$.

30. $xy - \overline{7 + xy}$.

Remove the brackets and simplify:

31. $27a^2 - [4 - 27a^2]$.

32. $35ab - [5 + 35ab]$.

33. $42xy - [-xy + 42]$.

34. $75x^2y^2 + [-x^2y^2 + 7]$.

35. $80mn + [75 - 80mn]$.

36. $[35x + 1] - 1$.

37. $[42 - x] + x$.

38. $-[27 + ab] - ab$.

Remove the parentheses and brackets and simplify:

39. $[a + b] - (a - b)$.

40. $[a - b] - (a + b)$.

41. $[a + b] + (-a - b)$.

42. $-[a + b] - (-a - b)$.

43. $-[a - b] - (-a + b)$.

44. $[x^2 + y^2 - z^2] - (x^2 - y^2 + z^2)$.

Exercise 62. Review of Chapter V

1. Add 7500 *rt*, 475 *rt*, 9650 *rt*, 8754 *rt*, 4965 *rt*.

2. In Ex. 1, if $r = 0.05$, and $t = \frac{1}{365}$, find the sum.

3. Representing money on which interest is allowed as positive; sums on which interest is charged as negative; the rate of interest allowed, by r; the rate of interest charged, by r'; one day in interest allowed, by $\frac{1}{365}$; one day in interest charged, by $\frac{1}{360}$; find the sum of the interest debits and credits on the following daily balances for 11 da.: + \$35,000, + \$37,500, − \$15,000, − \$125,000, + \$27,800, + \$250,000, − \$18,500, − \$30,000, − \$12,000, + \$96,000, + \$63,000.

4. Find the value of the last daily balance in Ex. 3 if it is given that $r = 0.02$, $r' = 0.04$.

5. If C = gross cost, O = overhead charges, P = profit, and S = selling price, write the meaning of the following formula:

$$C = S - (O + P).$$

6. Write the formula in Ex. 5 so that it shall have the same value, but without using the parentheses.

7. Add $O + P$ to each side of the formula derived in Ex. 6, and explain the meaning of the result in retail business.

8. Find the cost of an article which a dealer sells for \$38.13, if the overhead charges are 18% of the cost and the profit is 5% of the cost.

Add the following:

9.	10.	11.
4 gal. + 1 qt.	$720 \times 0.06 \times \frac{1}{365}$	£18 + 5*s*
9 gal. + 2 qt.	$4560 \times 0.06 \times \frac{1}{365}$	£19 + 7*s*
8 gal. − 1 qt.	$540 \times 0.06 \times \frac{1}{365}$	£ 5 − 3*s*
5 gal. + 1 qt.	$7200 \times 0.06 \times \frac{1}{365}$	£10 − 4*s*
6 gal. − 1 qt.	$673 \times 0.06 \times \frac{1}{365}$	£ 9 + 8*s*

Exercise 63. Review of Chapters I-V

1. Using the formula $i = prt$, find the interest on \$4000 for 35 da. at 5%, using 365 da. for 1 yr.

2. Find the cost of a piece of oak timber 18 ft. long, 6 in. wide, and 3 in. thick at 10¢ a board foot. Write the formula used.

3. Sheet copper that weighs 3 lb. to the square foot is to be used to cover a circular surface the radius of which is 5 ft. At 35¢ a pound how much will the copper cost?

4. A, B, and C have a partnership contract by which they agree to allow A and B \$1200 a year each for conducting the business, and to divide the remainder of the available funds by allowing 11 shares to A, 15 shares to B, and 20 shares to C. How should they divide \$11,600 of available funds?

5. Solve these equations: $3x + 0.5x = 70$; $\frac{2}{3}x = 15$.

6. I have 60 bu. of potatoes in barrels which hold 2 bu. each and cost 30¢ each. A dealer offers me \$35 for the lot. If I accept the offer, how much shall I realize per bushel?

7. Subtract $38x^2 - 5xy + 9y^2$ from $11x^2 + 11xy - 9y^2$.

8. In the following algebraic and numerical expressions

a	2456
b	3148
$a + b$	5604
$a + 2b$	8752
$2a + 3b$	14356

each number after the first two is derived by adding the two immediately above it. Continue this process till 10 lines are formed, and then add. What does the algebraic work show is always the relation of the sum of the ten addends to the seventh addend?

CHAPTER VI

MULTIPLICATION

Exercise 64. Multiplication of Monomials

Examples 1 to 23, oral

1. Multiply by 5: 2 mi.; 2 ft.; $2m$; $2f$; $2 \cdot 3$; $2x$.

2. Multiply by 7: 3 yd.; $3y$; 3 times a given number; $3n$.

3. Multiply by 9: 4 in.; $4i$; \$4; $4d$; $4¢$; $4c$; $4x^2y^2$; $4 \cdot 7$.

4. If the temperature is 3° below zero, what will it be when it is twice as much below zero?

5. How much is $2 \times (-3°)$? $2 \times (-4°)$? $2 \times (-3d)$?

State the following products:

6. $2 \times 12a$.	10. $3 \times 20x$.	14. $4 \times 50m$.
7. $2 \times 12ab$.	11. $3 \times 20xy$.	15. $4 \times 50m^3$.
8. $2 \times (-12)$.	12. $3 \times (-20)$.	16. $4 \times (-50)$.
9. $2 \times (-12a^2)$.	13. $3 \times (-20x^2)$.	17. $4 \times (-50\sqrt{a})$.

18. How may $a \times a$ be written more briefly? $a \times a \times a$? $a \times a^2$? $a^2 \times a$? $a \times a^3$? $a^2 \times a^2$? $a^3 \times a^2$?

19. State a rule for multiplying a^3 by a^4.

Find the following products:

20. $3 \times 4a^2$.	24. $x^2 \times 6x^3$.	28. $8 \times 36x^3$.
21. $a \times 4a^2$.	25. $5x^2 \times 6x^3$.	29. $8x \times 36x^3$.
22. $3a \times 4a^2$.	26. $7 \times 17m^2$.	30. $9x^2 \times 36x^2$.
23. $5 \times 6x^3$.	27. $7m \times 17m^2$.	31. $9x^2 \times 48x^3$.

82. Multiplication. Expressions like "2 times 3" show that the operation of multiplication had its origin in addition, 2×3 meaning $3 + 3$. When we come to extend the idea of multiplication to include negative and fractional multipliers, however, we have also to extend the meaning of the word "times," for it is just as impossible to add a number $2\frac{1}{2}$ times, -2 times, or $\frac{3}{4}$ times, as it is impossible to go to a store $2\frac{1}{2}$ times, -2 times, or $\frac{3}{4}$ times. So we give new meanings to the word "times" to cover cases like those mentioned. For example,

$2\frac{1}{2}$ times 6 means 2 times 6 plus $\frac{1}{2}$ of 6;

-2 times 3 means $0 - 3 - 3$, or -6;

$\frac{3}{4}$ times 12 means $\frac{3}{4}$ of 12, or $3 \times \frac{1}{4}$ of 12.

83. Laws of Signs. The case of -2×3 has already been explained on page 66. We may now state in different words the law of signs which is there given and which applies to all algebraic expressions:

In multiplication two like signs produce plus and two unlike signs produce minus.

That is, $+a \cdot (+b) = +ab$, $+a \cdot (-b) = -ab$,

$-a \cdot (-b) = +ab$, $-a \cdot (+b) = -ab$.

Evidently, therefore, *the product of an even number of negative factors is positive, and the product of an odd number of negative factors is negative.*

These laws lead to the consideration of powers of negative numbers. Thus,

$(-1)^2 = +1$, and $(-a)$ to any even power is positive;

$(-1)^3 = -1$, and $(-a)$ to any odd power is negative.

That is, *even powers of negative numbers are positive; odd powers of negative numbers are negative.*

84. Laws of Exponents. Since a^2 means $a \cdot a$, and a^3 means $a \cdot a \cdot a$, we see that, as we found in the exercise on page 93,

$$a^2 \cdot a^3 = a \cdot a \times a \cdot a \cdot a = a^5 = a^{2+3}.$$

Furthermore, $a^m = a$ taken m times as a factor

and $a^n = a$ taken n times as a factor.

Therefore $a^m \cdot a^n = a$ taken $m + n$ times as a factor.

That is, $$a^m \cdot a^n = a^{m+n}.$$

For example, $1.06^2 \times 1.06^3 = 1.06^5$, as we found on page 20. It should be observed, however, that 1.04^5 and 1.06^2 cannot be multiplied together in this way.

In multiplying monomials the exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.

Since $(a^2)^3$ means $a^2a^2a^2$, or a^6, and $(a^m)^n$ means $a^m a^m a^m \ldots$ to n of the factors a^m, therefore

$$(a^m)^n = a^{mn}.$$

For example, $(a^3)^5 = a^{15}$, $(x^7)^3 = x^{21}$, and so on.

85. Multiplication of Monomials. As we have seen in the exercise on page 93, to find the product of two monomials we may proceed as follows:

Find the product of the numerical coefficients, writing after this product the letters, each letter having an exponent equal to the sum of its exponents in the factors.

It will be found better to write the product in this order: the sign of the product, the product of the numerical coefficients, the letters in their alphabetical order, each with its proper exponent. For example,

$$3\,a^m b \cdot 4\,a^n b = 3 \cdot 4 \cdot a^m \cdot a^n \cdot b \cdot b = 12\,a^{m+n} b^2;$$

$$(-7\,x^7 y^8)^2 = -7 \cdot (-7) \cdot x^7 \cdot x^7 \cdot y^8 \cdot y^8 = 49\,x^{14} y^{12};$$

$$6\,abc \cdot (-7\,a^2 b^2 c^2) = 6 \cdot (-7) \cdot a \cdot a^2 \cdot b \cdot b^2 \cdot c \cdot c^2 = -42\,a^3 b^3 c^3.$$

In practice the result should usually be written rapidly, or stated orally, without the intermediate step given here.

Exercise 65. Multiplication of Monomials

Examples 1 to 42, oral

Perform the following multiplications:

1. a^2a^9.
2. b^3b^8.
3. c^9c^4.
4. d^7d^7.
5. e^9e^8.
6. m^3m^{11}.
7. p^4p^{12}.
8. $(a^{11})^3$.
9. $(a^{12})^2$.
10. $(a^ma)^3$.
11. $(a^ma^4)^2$.
12. $a^{3x}a^x$.
13. $a^{3x}a^{2x}$.
14. $ab \cdot ab$.
15. $a^2b \cdot a^2b$.
16. $a^2b^3 \cdot a^2b^3$.
17. $a^4b^7 \cdot a^5b^2$.
18. $p^9q^7 \cdot p^7q^9$.
19. $5\,a^7 \cdot 6\,a^5$.
20. $8\,(a^7)^4$.
21. $7\,(x^2)^3$.
22. $(8\,x^2)^2$.
23. $3\,a \cdot (-4\,a)$.
24. $3\,a \cdot (-4\,a^2)$.
25. $3\,a^m \cdot (-5\,a^n)$.
26. $ab \cdot (-ab)$.
27. $-ab \cdot ab$.
28. $-ab \cdot (-ab)$.
29. $x^2y^2 \cdot (-2\,xy)$.
30. $-3\,y^3 \cdot (-7\,y)$.
31. $(2\,ab^2c^3)^3$.
32. $(-2\,a^2b^3)^4$.
33. $(abc \cdot abc)^3$.

Express the following products as powers:

34. $2^5 \times 2^7$.
35. $3^4 \times 3^7$.
36. $1.06^2 \times 1.06^7$.
37. $1.03^8 \times 1.03^9$.
38. $(a+b)^4(a+b)^{10}$.
39. $20^2 \times 20 \times 20^5$.

40. If the circumference of a circle is πd, what is the sum of the circumferences of seven circles of diameter d?

41. If the circumference πd of a circle is multiplied by d, what is the result? What if it is multiplied by $\frac{1}{4}\,d$?

42. If the edge of a cube is $4\,x$, what is the volume?

Perform the following multiplications:

43. $-27\,x^m \cdot (-23\,x^7)$.
44. $-42\,x^{17} \cdot (-34\,x^9)$.
45. $-68\,p^m \cdot (-27\,p^n)$.
46. $37\,x^{4m} \cdot 67\,x^{9m}$.
47. $83\,x^{m+1} \cdot 75\,x^{m-1}$.
48. $43\,a^{p+4} \cdot 72\,a^{p-4}$.

86. Multiplication of a Polynomial by a Monomial. If we multiply 5 ft. 2 in. by 3, we have 15 ft. 6 in. Also, if we multiply 5 times one number plus 2 times another number by 3, we have 15 times the first plus 6 times the second. That is,

5 ft. 2 in.	$5f+2i$	$5x+2y$
3	3	3
15 ft. 6 in.	$15f+6i$	$15x+6y$

In the same way we have the following:

Operation	Check
$a^2 - 2ab + 3b^2$	$1-2+3=2$
ab	$1 \quad\quad =1$
$a^3b - 2a^2b^2 + 3ab^3$	$1-2+3=2$

In algebra it is more convenient to write the multiplier at the left.

In this check we let $a=1$ and $b=1$. This checks the coefficients, where the error is most liable to occur, but it does not check the exponents, since any power of 1 is 1. If a check upon the exponents is desired, let $a=2$ and $b=3$, or take any other convenient values.

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add these products.

Exercise 66. Multiplying by Monomials

Examples 1 to 7, oral

Perform the following multiplications:

1. $a(b+c)$.
2. $-a(b-c)$.
3. $2(2^2+2^3)$.
4. $2\pi r(r-1)$.
5. $a(b+c-d)$.
6. $-a(b-c+d)$.
7. $b^2(a^2-b^2+c^2)$.
8. $2x^2(x^2-2xy+y^2)$.
9. $4x^3(x^2-3x+27)$.
10. $-7ab(a^2-2ab+b^2)$.
11. $-9a^2b(a^2-2ab+b^2)$.
12. $4xy^2(2x^2-xy-3y^2)$.
13. $35a^3x^5(a^7-3a^4x^2+2ax^5)$.
14. $-8x^ny^n(x^ny^n-7xy-13)$.

87. Factors. Since $3x(3x+1) = 9x^2 + 3x$, we can see at once that the factors of $9x^2 + 3x$ are 3, x, and $3x+1$. When asked to factor $9x^2 + 3x$, we express the result thus:

$$9x^2 + 3x = 3x(3x+1).$$

Exercise 67. Products and Factors

Examples 1 to 6, oral

Express the following as algebraic sums of powers:

1. $a^2(a^2 + a^3 + a^4)$.
2. $x^3(x^2 - x^4 + x^7)$.
3. $5^3(5^7 + 5^3 + 5)$.
4. $7^3(7^8 - 7^6 + 7^4)$.
5. $4^{10}(4^6 + 4^3 + 4^2)$.
6. $z^{10}(z + z^{20} - z^{30})$.

Find the factors of the following binomials:

7. $a^2 + a^3$.
8. $x - x^2$.
9. $2x^2 - 2x^3$.
10. $3m^2 + 9m$.
11. $3m^2 - 9m$.
12. $3m^2 - 9m^3$.
13. $4x + 4y$.
14. $4x - 8y$.
15. $12x - 8y^2$.

Multiply the following:

16. $39(100 + 10 + 2)$.
17. $78(100 + 10 + 5)$.
18. $7(60 + 20 - 10)$.
19. $785(1000 - 1)$.
20. $45(9000 - 2)$.
21. $846(3000 + 100 + 2)$.

Find the products in Exs. 22, 23; the factors in Exs. 24, 25:

22. $x(x^3 + x^2 + x - 4)$.
23. $2m(m^2 + 3m - 4)$.
24. $x^4 + x^3 + x^2 - 4x$.
25. $2m^3 + 4m^2 - 8m$.

Find the factors of the following binomials:

26. $7x^2 + 14x$.
27. $7x^2 - 14x^3$.
28. $8a^3 + 16a$.
29. $\pi r^2 - 2r^2$.
30. $\pi r^2 - 2r$.
31. $\frac{1}{2}gt^2 - gt$.
32. $gt - \frac{1}{2}g$.
33. $p + prt$.
34. $\frac{1}{2}bh + \frac{1}{2}Bh$.
35. $\pi R^2 - \pi r^2$.
36. $2\pi R - 2\pi r$.
37. $Kx^2 + \frac{1}{2}x$.

88. Multiplication of a Polynomial by a Polynomial. If we multiply 43 by 21, we proceed as follows:

Multiplicand,	43	$40+3$
Multiplier,	21	$20+1$
Multiplying by 1 unit,	43	$40+3$
Multiplying by 2 tens,	86	$800+60$
Sum of partial products,	903	$800+100+3$

In a similar manner we multiply $4a^2-b$ by a^2+b, thus:

	Operation	Check
	$4a^2-b$	$4-1=3$
	a^2+b	$1+1=2$
Multiplying by a^2,	$4a^4-a^2b$	6
Multiplying by b,	$4a^2b-b^2$	
	$4a^4+3a^2b-b^2$	$4+3-1=6$

To multiply a polynomial by a polynomial, multiply the multiplicand by each term of the multiplier and add the partial products.

Exercise 68. Multiplying by a Binomial

Perform the following multiplications:

1. $(a+b)(a+2b)$.
2. $(a+b)(a-2b)$.
3. $(2a+b)(a-b)$.
4. $(2a-b)(a-b)$.
5. $(a+b)(a^2-b^2)$.
6. $(2xy-1)(2xy+1)$.
7. $(2xy+2)(2xy-2)$.
8. $(4x^2y+1)(4x^2y-3)$.
9. $(x^2y^2+1)(7x^2y^2-9)$.
10. $(a+b)(a-b)$.
11. $(a+b)(a+b)$.
12. $(a^2-b^2)(a-b)$.
13. $(a^2-b^2)(a^2+b^2)$.
14. $(abc-1)(a^2b^2c^2+5)$.
15. $(a^3b^2c-2)(a^3b^2c+2)$.
16. $(xy^2z^3+3)(xy^2z^3-3)$.
17. $(4a^2-5b^2)(3a^2+4b^2)$.
18. $(x^m+x^n)(5x^m-3x^n)$.

89. Arranging a Polynomial. If the exponents of a certain letter in the successive terms of a given polynomial decrease or increase from left to right the polynomial is said to be *arranged according to the descending powers* or *ascending powers* of that letter respectively.

Thus, $x^3 + 3x^2 - 4x + 1$ is arranged according to the descending powers of x, and $a^3 - 3a^2b + 3ab^2 - b^3$ is arranged according to the ascending powers of b and also according to the descending powers of a.

In multiplying, it simplifies the work greatly if both the polynomials are arranged according to the ascending or according to the descending powers of the same letter.

Multiply $4a^3 - 2a^2 + 7 - a$ by $3 + a^2 - 3a$.

Rearranging the polynomials, we multiply thus:

Operation	Check
$4a^3 - 2a^2 - a + 7$	$= 8$
$a^2 - 3a + 3$	$= 1$
$4a^5 - 2a^4 - a^3 + 7a^2$	8
$-12a^4 + 6a^3 + 3a^2 - 21a$	
$12a^3 - 6a^2 - 3a + 21$	
$4a^5 - 14a^4 + 17a^3 + 4a^2 - 24a + 21$	$= 8$

In the operation we multiply first by a^2, then by $-3a$, and finally by 3. We then add the partial products.

In the check we let $a = 1$. We have, then, only to add the coefficients, having $4 - 2 - 1 + 7 = 8$, and similarly for the others.

Multiply $x^2 - 3xy + 3y^2$ by $y^2 + x^2$.

Rearranging the multiplier, we multiply thus:

$x^2 - 3xy + 3y^2$	$= 1$
$x^2 + y^2$	$= 2$
$x^4 - 3x^3y + 3x^2y^2$	2
$x^2y^2 - 3xy^3 + 3y^4$	
$x^4 - 3x^3y + 4x^2y^2 - 3xy^3 + 3y^4$	$= 2$

Exercise 69. Multiplying by a Polynomial

Examples 1 to 4, oral

1. Arrange $a^2 + a^3 + 1 + a$ for multiplying.

2. Arrange $x + 1 + x^4 - x^2 + 3x^5 + x^6 + x^3 + x^7$ according to the ascending powers of x.

3. Arrange $x^2 + y^2 + 2xy$ according to the descending powers of x. How is it then arranged with respect to y?

4. Apply the check of $x = 1$, $y = 1$, and thus find whether $x^2 + 3xy + y^2$ can be the product of $x + y$ and $x + y$.

Multiply and check:

5. $(p + q)(p + q)$.
6. $(x + y)(x + y)$.
7. $(m + n)(m + n)$.
8. $(a + 2b)(a - 3b)$.
9. $(p + q)(p - q)$.
10. $(x + y)(x - y)$.

11. $a^2 - b^2$ by $a - b$; by $a + b$; by $a^2 + b^2$.

12. $a^2 + 2ab + b^2$ by $a + b$; by $a - b$; by $a^2 + 2ab + b^2$.

13. $a^2 - 2ab + b^2$ by $a - b$; by $a + b$; by $a^2 - 2ab + b^2$.

14. $7h + 4t + 3u$ by $5t + 7u$.

Teachers should call attention to the relation of Ex. 14 to 57×743.

15. $7h + 6t + 2u$ by $5h + 7u$; by $5h + 3t + 7u$.

16. $4x^3 + 2x^2y - 3xy^2 + 2y^3$ by $x^2 - 2xy + 4y^2$.

17. $3m^3 + 2m^2n - 4mn^2 + n^3$ by $m^2 + n^2$.

18. $5m^4 + 2m^2 + m + 1$ by $m^2 + 2m - 1$.

19. $3p^3 + 2p^2 + p - 4$ by $p^2 - 2p + 3$.

20. $4x^3 - 3x^2 + 2x - 3$ by $x^2 - 2x - 5$.

21. $a^2b^2 + 3ab - 2$ by $a^2b^2 - 3ab + 2$.

22. $a^2 + x^2 - 3ax + 2$ by $a^2 - x^2 + 3ax - 2$.

23. $x^3 + y^3 - x^2y + xy^2$ by $x^2 - xy + y^2$.

90. Check of Nines. One of the first things to learn in commercial work is the necessity for absolute accuracy. Speed is desirable, but accuracy is absolutely necessary. To insure this accuracy various checks on the number work are used, and among them is the check of nines. This check is used chiefly in multiplication and division, and we shall first give the method, using algebra to explain it later.

To find the remainder in the division of a number by 9, add the digits of the number, casting out the 9's as they appear.

For example, to find the remainder in the division of 1,478,603 by 9, add the digits and cast out the 9's, thus: $1 + 4 + 7 = 12$; cast out the 9 that is in 12 and there remains 3; then $3 + 8 = 11$, from which we cast out 9 and there remains 2; then $2 + 6 + 3 = 11$, from which we cast out 9 and there remains 2. If we divide 1,478,603 by 9, we shall find that the remainder is 2, as here shown.

Practically, it is not necessary to add all the digits, for we can easily see that $1 + 8 = 9$ and $6 + 3 = 9$, so we need only cast out the 9 in $4 + 7$.

The remainder arising from dividing a number by 9, or from dividing the sum of the digits by 9, is called the *excess of nines* in the number.

Thus 2 is the excess of 9's in 1,478,603.

Multiplication may be checked by finding the excess of nines in a product and in each factor, the excess in the product being equal to the excess in the product of the excesses in the factors.

For example, consider the case of $38 \times 27{,}394$. Here the excess in 27,394 is 7; that in 38 is 2; and that in the product is 5. If the multiplication is correct, the excess in 2×7 must be 5. This is the case, because the excess in 14 is 5.

27394	Excess = 7
38	Excess = 2
219152	14
82182	
1040972	Excess = 5

The check of nines is only a check, not an absolute proof.

Exact division may be checked by the check of multiplication, because the dividend may be taken as the product, and the divisor and the quotient may be taken as the factors.

For example, in the case of $1740 \div 12 = 145$, we see that $1740 = 12 \times 145$. Taking the excesses of 9's, we see that in 1740 it is 3, in 12 it is 3, and in 145 it is 1. Furthermore, 3 (excess in 1740) $= 3 \times 1$ (product of excesses in 12 and 145).

$$12\overline{)1740}\quad 145$$

If a division is not exact, find the product of the excess in the divisor by the excess in the quotient, and add the excess in the remainder. The sum, or the excess in the sum, should be equal to the excess in the dividend.

For example, in the case of $118{,}784 \div 4892$, we see that the quotient is 24 and the remainder is 1376. Taking the excesses of 9's, we see that in 118,784 it is 2, in 4892 it is 5, in 24 it is 6, and in 1376 it is 8. We can now check the result as follows:

Excess in 4892 = 5
Excess in 24 = 6
Product of excesses = 30, excess = 3
Excess in 1376 = 8
Sum of excesses = 11, excess = 2
Excess in 118,784 = 2

$$\begin{array}{r} 24 \\ 4892\overline{)118784} \\ 9784 \\ \hline 20944 \\ 19568 \\ \hline 1376 \end{array}$$

Exercise 70. Check of Nines

Perform the following operations and check the work:

1. 275×398.
2. 309×588.
3. 644×966.
4. 722×880.
5. 4.29×6.77.
6. 5.39×7268.
7. $68{,}781 \div 51$.
8. $54{,}227 \div 83$.
9. $52{,}870 \div 85$.
10. $68{,}725 \div 77$.
11. $87{,}316 \div 57$.
12. $46{,}817 \div 67$.
13. 6296×5287.
14. 5865×7963.
15. 8559×6279.
16. $22{,}712 \div 641$.
17. $38{,}499 \div 827$.
18. $86{,}225 \div 653$.

91. Proof of Check of Nines. Algebra allows us to prove the statements on pages 102 and 103.

Any number of three figures may be represented by $100\,a + 10\,b + c$, because a, b, and c may represent any digits.

For example, 874 may be represented by $100\,a + 10\,b + c$ by letting $a = 8$, $b = 7$, $c = 4$.

But $100\,a + 10\,b + c = (99\,a + a) + (9\,b + b) + c$, and this may be written $(99\,a + 9\,b) + (a + b + c)$, where every part of the first parenthesis is divisible by 9. Therefore the only remainder possible comes from dividing $a + b + c$ (the sum of the digits) by 9.

For example, $874 = 800 + 70 + 4 = 8 \times 99 + 7 \times 9 + 8 + 7 + 4$. Hence the only remainder we can obtain is the remainder which comes from dividing $8 + 7 + 4$ (the sum of the digits) by 9.

The same result as to divisibility by 9 is evidently obtained if we take a number of four or more figures.

It is evident that any number is equal to some multiple of 9 plus the excess of nines. That is, $58 = 9 \times 6 + 4$, $343 = 9 \times 38 + 1$, and in general every number is of the form $9\,m + e$.

If we represent two numbers by $9\,m + e$ and $9\,m' + e'$, we find that their product is $9^2 mm' + 9(me' + m'e) + ee'$.

The teacher should explain that $9\,m + e$ means any multiple of 9 plus the excess, and that $9\,m' + e'$ means any other multiple of 9 plus the excess; that is, that $9\,m + e$ and $9\,m' + e'$ represent any two numbers whatever.

$$\begin{array}{r} 9\,m + e \\ 9\,m' + e' \\ \hline 9^2 mm' + 9(me' + m'e) + ee' \end{array}$$

This product is evidently divisible by 9, except the last term, ee', which is the product of the excesses. That is, as stated on page 102, the *excess of 9's in the product is equal to the excess in the product of the excesses of 9's in the factors.*

92. Special Products. There are certain products which are required so often in algebra that it is helpful to be able to write them at once without the labor of multiplying.

93. Square of the Sum or Difference of Two Numbers. If we multiply $a + b$ by $a + b$, the product is $a^2 + 2\,ab + b^2$; and if we multiply $a - b$ by $a - b$, the product is $a^2 - 2\,ab + b^2$.

$$\begin{array}{rr} a + & b \\ a + & b \\ \hline a^2 + & ab \\ & ab + b^2 \\ \hline a^2 + 2\,ab + & b^2 \end{array} \qquad \begin{array}{rr} a - & b \\ a - & b \\ \hline a^2 - & ab \\ - & ab + b^2 \\ \hline a^2 - 2\,ab + & b^2 \end{array}$$

The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.

That is, $$(a+b)^2 = a^2 + 2\,ab + b^2.$$

The expression $a^2 + 2\,ab + b^2$ is called the *expansion* of $(a + b)^2$. For example,

$$13^2 = (10 + 3)^2 = 10^2 + 2 \times 10 \times 3 + 3^2 = 169;$$
$$35^2 = (30 + 5)^2 = 30^2 + 2 \times 30 \times 5 + 5^2 = 1225;$$
$$125^2 = (100 + 25)^2 = 100^2 + 2 \times 100 \times 25 + 25^2$$
$$= 10{,}000 + 5000 + 625 = 15{,}625.$$

ab	b^2
a^2	ab

a b

Thus we see that we can mentally square certain numbers, saving ourselves the trouble of writing out the work.

It is easily seen that the figure representing the square on $a + b$ is made up of a^2, ab, ab, b^2, and therefore is $a^2 + 2\,ab + b^2$.

The square of the difference of two numbers is the square of the first, minus twice their product, plus the square of the second.

That is, $$(a-b)^2 = a^2 - 2\,ab + b^2.$$

Therefore $$37^2 = (40 - 3)^2 = 40^2 - 2 \times 40 \times 3 + 3^2 = 1369.$$

Of course we might also square 37 in this way:

$$37^2 = (30 + 7)^2 = 900 + 2 \times 30 \times 7 + 49 = 1369.$$

Exercise 71. Square of the Sum or the Difference

Examples 1 to 37, oral

Expand the following:

1. $(x+y)^2$.	10. $(x-y)^2$.	19. $(2a+b)^2$.
2. $(p+q)^2$.	11. $(p-q)^2$.	20. $(2a-b)^2$.
3. $(m+n)^2$.	12. $(m-n)^2$.	21. $(2a+1)^2$.
4. $(a+x)^2$.	13. $(a-x)^2$.	22. $(2a-1)^2$.
5. $(b+y)^2$.	14. $(b-2y)^2$.	23. $(4x+y)^2$.
6. $(a+1)^2$.	15. $(a-2)^2$.	24. $(5x-y)^2$.
7. $(a+2)^2$.	16. $(a-7)^2$.	25. $(x+6y)^2$.
8. $(x+3)^2$.	17. $(x-6)^2$.	26. $(x-7y)^2$.
9. $(4+n)^2$.	18. $(8-n)^2$.	27. $(2a+3)^2$.

28. How would you proceed if you wished to write the square of $5x+3y$ without multiplying in the usual way?

Expand the following:

29. 11^2.	38. 62^2.	47. 102^2.	56. $(7x^3+6)^2$.
30. 12^2.	39. 71^2.	48. 503^2.	57. $(7x^3-6)^2$.
31. 21^2.	40. 88^2.	49. 902^2.	58. $(5x^2+2y^2)^2$.
32. 22^2.	41. 92^2.	50. 998^2.	59. $(5x^2-2y^2)^2$.
33. 16^2.	42. 98^2.	51. 5006^2.	60. $(a^2b^2c^2+12)^2$.
34. 31^2.	43. 79^2.	52. 4994^2.	61. $(a^2b^2c^2-12)^2$.
35. 14^2.	44. 78^2.	53. 5005^2.	62. $(x^4y^4+3n)^2$.
36. 41^2.	45. 91^2.	54. 7007^2.	63. $(x^4y^4-3n)^2$.
37. 51^2.	46. 52^2.	55. 1003^2.	64. $(7a^3b^2c+11)^2$.

65. Square $(a+b)+c$ as if $a+b$ were one term and then remove the parentheses in the result.

94. Product of the Sum and the Difference of Two Numbers. If we multiply $a - b$ by $a + b$, or $a + b$ by $a - b$, the product is $a^2 - b^2$, as here shown:

$$\begin{array}{r} a - b \\ a + b \\ \hline a^2 - ab \\ \quad ab - b^2 \\ \hline a^2 \quad - b^2 \end{array} \qquad \begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ \quad - ab - b^2 \\ \hline a^2 \quad - b^2 \end{array}$$

Therefore *the product of the sum and the difference of two numbers is the difference of their squares.*

That is, $$(a + b)(a - b) = a^2 - b^2.$$

For example, $(5a + 3b)(5a - 3b) = 25a^2 - 9b^2$,

and $33 \times 27 = (30 + 3)(30 - 3) = 900 - 9 = 891.$

Exercise 72. Product of the Sum and the Difference

Examples 1 to 13, oral

Multiply as indicated:

1. $(x + y)(x - y)$.
2. $(a + 7)(a - 7)$.
3. $(3x + 8)(8 - 3x)$.
4. $(p^4 + 3)(p^4 - 3)$.
5. $(2x^2 + 1)(2x^2 - 1)$.
6. $(5x^7 + 1)(1 - 5x^7)$.
7. $(4a^2 + 2)(4a^2 - 2)$.
8. $(a^m b^m + 1)(a^m b^m - 1)$.

Find the products of the following:

9. 32×28.
10. 34×26.
11. 41×39.
12. 42×38.
13. 51×49.
14. 61×59.
15. 92×88.
16. 52×68.
17. 29×31.
18. 44×56.
19. 99×101.
20. 98×102.
21. 117×123.
22. 994×1006.
23. 2493×2507.

95. Factoring the Difference of Two Squares. Reversing the work on page 107, we see that

The difference of the squares of two numbers is the product of the sum and the difference of the numbers.

That is, $a^2 - b^2 = (a + b)(a - b)$.

Factor the binomials $a^4 - b^2$ and $x^8 - y^8$.

$$a^4 - b^2 = (a^2 + b)(a^2 - b).$$
$$x^8 - y^8 = (x^4 + y^4)(x^4 - y^4)$$
$$= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2)$$
$$= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y).$$

Exercise 73. Factoring the Difference of Two Squares

Examples 1 to 15, oral

Factor the following expressions:

1. $a^2 - x^2$.	**6.** $8^2 - 7^2$.	**11.** $25^2 - a^2$.
2. $x^2 - a^2$.	**7.** $8^2 - a^2$.	**12.** $x^2 - 10^2$.
3. $a^2 - 1$.	**8.** $9^2 - 5^2$.	**13.** $4a^2 - b^2$.
4. $m^2 - 4$.	**9.** $49^2 - 1$.	**14.** $x^2 - 16$.
5. $n^2 - 9$.	**10.** $75^2 - 15^2$.	**15.** $a^2b^2 - c^2$.

Factor the following numbers:

16. 143. **17.** 120. **18.** 80. **19.** 624. **20.** 9999.

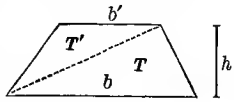
In Ex. 16, $143 = 144 - 1 = 12^2 - 1$; in Ex. 19, $624 = 25^2 - 1$.

Factor the following expressions:

21. $x^4 - 16y^4$.	**25.** $x^2 - 0.81y^2$.	**29.** $(x + y)^2 - 4z^2$.
22. $1 - x^8y^8$.	**26.** $81x^4 - 1$.	**30.** $(x - y)^2 - z^2$.
23. $x^4y^4 - 1$.	**27.** $a^4b^4c^4 - d^4$.	**31.** $(a - b)^2 - (c - d)^2$.
24. $x^8y^8 - a^8b^8$.	**28.** $a^8b^8c^8 - d^8$.	**32.** $(p - 1)^2 - (r - 1)^2$.

Exercise 74. Applications of Factoring

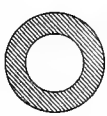
1. In the trapezoid here shown, the area of triangle T is $\frac{1}{2}bh$, and the area of triangle T' is $\frac{1}{2}b'h$. What is the area of the trapezoid? Factor the result, letting $\frac{1}{2}h$ be one of the factors.



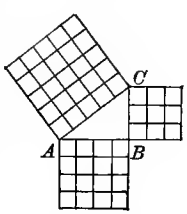
2. In a cylinder the area of each base is πr^2, and the area of the curved surface is $2\pi rh$. This makes the total surface how much? Factor the result, thus simplifying the formula.



3. The area of the outside circle here shown is πa^2, and the area of the inside circle is πb^2. What is the area of the ring formed by the two circles? Factor the result, thus simplifying the formula.

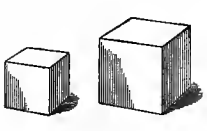


4. In the figure here shown we have found that if $AB = b$, $BC = a$, and $AC = h$, then $h^2 = a^2 + b^2$, or $b^2 = h^2 - a^2$. Write the equation $b^2 = h^2 - a^2$ with the second member factored.



5. Using the factored form found in Ex. 4, find the value of b^2 when $h = 5$, $a = 3$; when $h = 5$, $a = 4$; when $h = 35$, $a = 28$; when $h = 45$, $a = 36$.

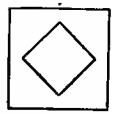
6. The edge of the larger of two cubes is a, and that of the smaller is b. The area of the base of one is how much greater than the area of the base of the other? Factor the result and evaluate for $a = 30$, $b = 20$.



7. In Ex. 6 what is the area of the entire surface of the six faces of each cube? What is the difference in area? Factor the result and evaluate for $a = 30$, $b = 20$. Evaluate the result also for $a = 40$, $b = 10$.

8. The amount of principal and interest on a note is given by the formula $a = p + prt$. Factor the second member of the equation and evaluate for $p = 350$, $r = 6\%$, $t = 3$.

9. From a square of side a is cut a square of side b, as shown in the figure. What is the area of the remaining part? Factor the result and evaluate for $a = 60$, $b = 20$.



10. The area of the surface of a sphere of radius r is $4\pi r^2$. What is the difference between the surfaces of two spheres of radius a and b respectively? Factor the result and evaluate for $a = 5$, $b = 4$, and $\pi = 3\frac{1}{7}$.

Factor the following expressions:

11. $(2x + 3y)^2 - a^2$.

12. $(3a - 2b)^2 - y^2$.

13. $(4m + n)^2 - x^2$.

14. $a^2 - (p - q)^2$.

15. $9a^2 - (a + b)^2$.

16. $16b^2 - (c - d)^2$.

17. $(x + y)^2 - (a - b)^2$.

18. $(x - y)^2 - (a + b)^2$.

19. $(x - y)^2 - (a - b)^2$.

20. $(2a + 1)^2 - (2b + 1)^2$.

21. $(1 - a)^2 - (5a - b)^2$.

22. $(a + 2b)^2 - (p - q)^2$.

23. Find by the formula of § 95 the value of $a^2 - b^2$ when $a = 37.1$, $b = 15.3$.

24. By actual multiplication find the values of 37.1^2 and 15.3^2, and then find the difference between the two results.

Exs. 23 and 24 show the superiority of algebra over arithmetic.

25. Multiply 1002 by 998. Also find orally the product of $1000 + 2$ and $1000 - 2$.

26. One square tile is 8.8 in. on a side and another is 7.2 in. on a side. What is the difference in area of the tiles?

27. Find the difference in area of two circles of radius 7.3 in. and 5.7 in. respectively.

96. Product of Two Binomials. If two binomials have a common term, their product can easily be written. For example,

$$\begin{array}{r} x+7 \\ x+5 \\ \hline x^2+7x \\ \quad 5x+35 \\ \hline x^2+12x+35 \end{array} \qquad \begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ \quad bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

The product of two binomials having a common term is equal to the square of the common term, plus the product of the common term by the algebraic sum of the other terms, plus the product of the other terms.

That is, $(x+a)(x+b)=x^2+(a+b)x+ab.$

Thus $(x+7)(x-3)=x^2+4x-21,$

because $+7+(-3)=4$, and $+7\times(-3)=-21.$

Similarly, $(x-9)(x-6)=x^2-15x+54,$

and $(a+7)(a+7)=a^2+14a+49.$

Exercise 75. Product of Two Binomials

All work oral

Multiply as indicated:

1. $(a+2)(a+3)$.	10. $(a+2)(a-3)$.	19. $(a-2)(a-3)$.
2. $(a+5)(a+7)$.	11. $(a-5)(a+7)$.	20. $(a-5)(a-7)$.
3. $(x+9)(x+3)$.	12. $(x+9)(x-3)$.	21. $(x-9)(x+1)$.
4. $(a+4)(a+5)$.	13. $(a-4)(a+5)$.	22. $(a-4)(a-3)$.
5. $(a+7)(a+3)$.	14. $(a+7)(a-3)$.	23. $(a-7)(a+3)$.
6. $(p+9)(p+4)$.	15. $(p-9)(p+4)$.	24. $(p-9)(p-4)$.
7. $(x+6)(x+2)$.	16. $(x+6)(x-2)$.	25. $(x-6)(x+4)$.
8. $(x+5)(x+9)$.	17. $(x-5)(x+9)$.	26. $(x-5)(x-9)$.
9. $(x+2)(x+7)$.	18. $(x+2)(x-7)$.	27. $(x-2)(x+7)$.

97. Application of Products. When two numbers differ but little from a common number whose square is easily found mentally, as in the case of 105 and 103, their product can be found mentally. Consider the following:

$x + 5$	$100 + 5$
$x + 3$	$100 + 3$
$x^2 + 5x$	$10{,}000 + 5 \times 100$
$3x + 15$	$3 \times 100 + 15$
$x^2 + 8x + 15$	$10{,}000 + 8 \times 100 + 15$

Practically, therefore, to multiply 105 by 103 we simply think of $10{,}000 + 800 + 15$, or 10,815.

Exercise 76. Products of Numbers

All work oral

1. What is the cost of 31 yd. of dimity at 32¢ a yard?

Think of 31×32 as $(30 + 1)(30 + 2)$, or $900 + 3 \times 30 + 2$, or 992.

2. What is the cost of 45 lb. of butter at 42¢ a pound?

3. If I pay 85% of 82% of the list price of an article, what per cent of the list price do I pay?

4. A grocer purchased 42 crates of eggs, each containing 4 doz. How many eggs did he purchase?

State the following products:

5. 84×88.	12. 71×73	19. 102×103.
6. 43×44.	13. 83×84.	20. 107×109.
7. 52×57.	14. 61×64.	21. 106×108.
8. 21×24.	15. 51×58.	22. 203×204.
9. 92×92.	16. 91×92.	23. 302×303.
10. 90×96.	17. 87×82.	24. 302×304.
11. 82×83.	18. 95×92.	25. 501×502.

98. Further Applications of Formulas. The simple formula

$$a(b - c) = ab - ac$$

has an important application in computation, as in the case of 945×998. From the above formula we see that

$$945 \times 998 = 945 \times (1000 - 2) = 945{,}000 - 1890.$$

That is, in this case we simply annex three zeros to 945, and from the result we subtract 2×945, obtaining 943,110. This is simpler than to multiply 998 by 945 in the usual way.

This method will be found to be of advantage when $b = 100, 1000, \cdots$, and c is a small number.

945,000
1,890
943,110

A similar application of algebra to commercial computations may be seen in the case of the formula

$$a(cx + c) = acx + ac.$$

If, for example, we wish to multiply 586 by 246, we may note that $246 = 6 \times 40 + 6$, and hence that we may multiply 586 by 6, and then multiply this product by 40 and add the results.

Here $a = 586$, $c = 6$, and $x = 40$, so that $ac = 3516$, $acx = 140{,}640$, and the product is $acx + ac$, or 144,156.

586
246
3516
14064
144156

Exercise 77. Products of Numbers

Multiply as indicated:

1. 97×845.
2. 98×945.
3. 99×784.
4. 96×856.
5. 94×973.
6. 99×877.
7. 9970×6487.
8. 9995×5485.
9. 9990×8456.
10. 9980×7565.
11. 9998×8767.
12. 9997×7564.
13. 426×486.
14. 936×785.
15. 728×943.
16. 355×658.
17. $972 \times \$5.98$.
18. $357 \times \$52.40$.

99. Further Applications of Formulas. Since the product is the same, however we group the factors, we know that

$$abc = (ab)c = a(bc) = b(ac).$$

If, then, we wish to find the product of 684, $\frac{1}{4}$, and 100, we may multiply 684 by $\frac{1}{4}$ of 100, or we may just as well multiply 684 by 100 and then by $\frac{1}{4}$.

This is the principle of *aliquot parts*, 25 being an aliquot part of 100. That is, if the multiplier can be expressed as an easy fractional part of 10, of 100, of 1000, $\cdots$, we may first multiply by 10, 100, 1000, $\cdots$, as the case may be, and then by the fraction.

Thus, in the case of 25×684, we have

$$25 \times 684 = \tfrac{1}{4} \text{ of } 100 \times 684 = \tfrac{1}{4} \text{ of } 68{,}400 = 17{,}100.$$

Exercise 78. Products of Numbers

Examples 1 to 12, oral

Multiply as indicated:

1. 50×784.
2. 25×680.
3. 75×888.
4. $33\frac{1}{3} \times 960$.
5. $37\frac{1}{2} \times 880$.
6. $14\frac{2}{7} \times 280$.

Find the cost of each of the following:

7. 32 yd. at $37\frac{1}{2}$¢.
8. 84 yd. at $62\frac{1}{2}$¢.
9. 96 yd. at $87\frac{1}{2}$¢.
10. 96 T. at \$6.25.
11. 80 T. at \$3.75.
12. 8.8 T. at \$6.25.

Multiply as indicated:

13. $309 \times 333\frac{1}{3}$.
14. $28\frac{4}{7} \times 280$.
15. 250×536.
16. 785×25¢.
17. 125×9664.
18. $0.83\frac{1}{3} \times 7548$.
19. $0.187\frac{1}{2} \times 7564$.
20. 1875×95.75.
21. $1250 \times \$40.56$.
22. $750 \times \$94.40$.
23. $625 \times \$140.80$.
24. 0.0625×840.

100. Further Applications of Formulas. Since the product is the same, whatever the order of the factors, we know that

$$ab = ba,$$

and hence that 89% of \$25 is the same in value as 25% of \$89, which is equal to $\frac{1}{4}$ of \$89. The product $\frac{1}{4}$ of \$89 is more easily found than the product 89% of \$25.

In commercial work the student should take advantage of this important principle, since he will thereby save himself a great deal of labor. It is a waste of time to find, for example, 96% of \$625 when we need only take $\frac{5}{8}$ of \$960, or $5 \times$ \$120, or $\frac{1}{2}$ of $10 \times$ \$120, or $\frac{1}{2}$ of \$1200.

Consider the case of 88% of \$625. This is equal to $62\frac{1}{2}$% of \$880, or $\frac{5}{8}$ of \$880. If the position of the decimal point is not clear, this example may be written

$$0.88 \times \$625 = 0.880 \times \$625 = 0.625 \times \$880 = \tfrac{5}{8} \text{ of } \$880.$$

That is, *the decimal point should be moved as many places to the left in one factor as it is moved to the right in the other.*

Exercise 79. Products of Numbers

All work oral

State the value of each of the following:

1. 17% of \$50.
2. 16% of \$125.
3. 24% of \$125.
4. 88% of \$750.
5. 92% of \$7500.
6. 45% of \$5000.
7. 106% of \$250.
8. 16% of \$2500.
9. 104% of \$250.
10. 108% of \$250.
11. 18.6% of \$50.
12. 19.6% of \$25.
13. 8% of \$1875.
14. 32% of \$125.
15. 64% of \$1250.
16. 24% of \$625.
17. 112% of \$250.
18. 104% of \$250.
19. 68% of \$1250.
20. 77% of \$5000.
21. 18% of \$12.50.
22. 29% of \$12.50.
23. 8% of \$8750.
24. 16% of \$6250.

101. Arrangement of Partial Products. The student may have wondered why, in algebra, we arrange the partial products by multiplying from left to right instead of from right to left as in arithmetic.

Our work in algebra shows us that it makes no difference, since $(a+b)c = ac+bc = bc+ac$. That is, we may have any of these arrangements:

$2t+7$	27	$2t+7$	27
$t+3$	13	$t+3$	13
$6t+21$	81	$2t^2+7t$	27
$2t^2+7t$	27	$6t+21$	81
$2t^2+13t+21$	351	$2t^2+13t+21$	351

Here $t=10$, so that $2t+7=27$ and $t+3=13$. We see that the result is the same whatever the order of multiplying, and it is merely a matter of habit and convenience as to which order we take. In commercial work, computers commonly begin with the left-hand figure, a plan especially valuable when unnecessary decimals are to be eliminated.

Exercise 80. Arranging Partial Products

1. Multiply $3h+2t+7$ by $2h+3t+5$ and multiply 327 by 235, arranging the partial products in different orders.

As in Ex. 1, perform the following multiplications:

2. $(7t+8)(8t+2)$ and 78×82.

3. $(5h+6)(7h+4)$ and 506×704.

4. $(5h+2t+3)(8h+7t+5)$ and 523×875.

5. $(7t^3+8t^2+4t+3)(8t^3+7t^2+9t+7)$ and 7843×8797.

6. Consider algebraically the case of multiplying 4575 by 3600, finding why it is that the zeros may be neglected in the multiplication if they are annexed to the product.

Exercise 81. Review of Chapter VI

Examples 1 to 14, oral

State the following products:

1. 79×81.
2. 58×62.
3. 71×72.
4. 81×84.
5. 85×85.
6. 41×41.
7. 480×998.
8. 565×999.

Express as powers the following products:

9. $a^5 \times a^7$.
10. $x^6 \times x^8$.
11. $35^2 \times 35^3$.
12. $1.06^4 \times 1.06^8$.
13. $(1+r)^6 \times (1+r)^7$.
14. $1.04^6 \times 1.04^7$.

15. Multiply $8x^3 + 5x^2 - 7$ by $15x^4$.

16. Multiply $9a^2 + 2ab - 3b^2$ by $4a^2 - 5ab + 6b^2$.

17. Multiply $p(1+r)^5$ by $(1+r)^7$; $p(1+r)^8$ by $(1+r)^3$.

18. Multiply $878 \times (10{,}000 - 2)$; $5765 \times (10{,}000 + 100 + 3)$.

19. The amount of a sum at simple interest is expressed by the equation $a = p + prt$. Factor the second member.

20. Prove that the excess of 9's in the product of any two numbers is the same as the excess of 9's in the product of the excesses of the numbers.

Write the factors of the following:

21. $x^2 - 64$.
22. $16m^2 - n^2$.
23. $x^2 - 81$.
24. $x^4 - 1$.
25. $x^2 - 225$.
26. $169x^2 - 4y^2$.

27. Using the short methods of this chapter, extend the following items and find the total of each column:

42 yd. at 25¢			375 lb. at $1.60		
37 lb. at 31¢			46 bbl. at $1.25		
51 bbl. at $51			18 T. at $6.25		
19 yd. at 25¢			96 T. at $8.75		

Exercise 82. Review of Chapters I–VI

1. Find the amount of \$9000 for 4 yr. at 6%, the interest being compounded annually.

2. Two concentric circles have radii r and r' respectively. Derive an expression for the area between the circumferences, and factor this expression.

3. By the aid of an equation find the sum of which \$575 is 18%; the sum of which \$675 is 27%.

4. Six coal companies enter into an agreement to sell only through a central selling agency. The "equalizer" is to apportion the orders to A, B, C, D, E, and F in proportion to 27, 20, 13, 13, 6, 5. What will be each company's share of an order for 1680 T. of coal?

5. Add the following and explain the algebraic principle which is involved: $\$800 \times 0.06 \times \frac{1}{365}$, $\$1950 \times 0.06 \times \frac{1}{365}$, $\$2400 \times 0.06 \times \frac{1}{365}$, $\$5400 \times 0.06 \times \frac{1}{365}$, $\$750 \times 0.06 \times \frac{1}{365}$, $\$950 \times 0.06 \times \frac{1}{365}$.

6. Using negative numbers when convenient in shortening the work, add the following: 31 gal. 1 qt., 52 gal. 3 qt., 18 gal. 1 qt., 15 gal. 3 qt., 16 gal. 2 qt., 18 gal. 3 qt.

7. Using the formula for the volume of a rectangular solid, $v = bh$, where b = area of base and h = height, find the cost of cutting and storing $\frac{1}{2}$ A. of ice that is 18 in. thick, if 1 cu. ft. of ice weighs 60 lb. and the cost is 30¢ a ton.

Remember that 1 A. = 160 sq. rd., and 1 rd. = $16\frac{1}{2}$ ft.

8. Multiply $8x^3 + 3x^2 - 5x + 7$ by $2x^2 - 3x + 4$.

9. Factor: $121p^4 - 1$; $49x^2y^2 - 121$; $4x^2 - 25$; $x^2 - 144$.

10. Factor: $p + prt$; $gt^2 + gt$; $4\pi r^2 - 4\pi r'^2$.

11. Using the first result in Ex. 10, find the amount of \$900 at simple interest for 11 yr. at 4%.

CHAPTER VII

DIVISION

Exercise 83. Division of Monomials

All work oral

1. Divide by 2: 4 mi.; $4m$; $4 \cdot 5$; $4x$; $4 \cdot 3 \cdot 5$; $4xy$.

2. Divide by 5: 10 lb.; $10l$; $10n$; $10n^2$; $10xy$; $10abc$.

3. Divide by 9: 27 rd.; $27r$; 36 in.; $36i$; 45 acres; $45a$.

4. If the temperature is 8° below zero, what will it be when it is half as much below zero? How much is $-8° \div 2$?

Perform the following divisions:

5. $64¢ \div 8$.
6. $64c \div 8$.
7. $64xy \div 8$.
8. $64\sqrt{a} \div 8$.
9. $-4° \div 2$.
10. $-4x \div 2$.
11. $-4abc \div 2$.
12. $-16x^2 \div 2$.
13. $75ab \div 25$.
14. $60a^2x \div 30$.
15. $45p^2q^2 \div 15$.
16. $90\sqrt{x} \div 45$.

17. How do you divide a monomial by a positive integer?

18. Divide \$8 by \$2; $8d$ by $2d$; 8 ft. by 2 ft.; $8f$ by $2f$.

Divide the following:

19. $8x$ by $2x$.
20. $10x$ by $2x$.
21. $15y$ by $5y$.
22. $16z$ by $8z$.
23. $25m$ by $5m$.
24. $36p$ by $4p$.
25. $40a$ by $10a$.
26. $75b$ by $25b$.
27. $125a^2$ by $5a^2$.
28. $275x^2$ by $25x^2$.
29. $250p^2$ by $50p^2$.
30. $175m$ by $25m$.

102. Law of Exponents. Since division is the inverse of multiplication, we have the following:

From $a^2 \cdot a^3 = a^5$ we have $a^5 \div a^3 = a^2$,

and from $a^x \cdot a^y = a^{x+y}$ we have $a^{x+y} \div a^y = a^x$;

or, in general $a^m \div a^n = a^{m-n}$.

The exponent of any factor of the quotient is equal to the exponent of that factor of the dividend minus the exponent of that factor of the divisor.

For example, $-21\,x^9y^4 \div 7\,x^5y = -3\,x^4y^3$.

In algebra, division is usually expressed in the fractional form, thus:

$$\frac{48\,a^5b^4c^3}{-4\,abc} = -12\,a^4b^3c^2.$$

103. Law of Signs. The law of signs, page 67, is as follows:

In division two like signs produce plus; two unlike signs produce minus.

That is, $\frac{4}{2} = 2$, $\frac{-4}{-2} = 2$, $\frac{-4}{2} = -2$, and $\frac{4}{-2} = -2$.

Exercise 84. Division of Monomials

Examples 1 to 5, oral

Perform the indicated divisions:

1. $\frac{3^5}{3^2}$.
2. $\frac{m^3n^4}{mn}$.
3. $\frac{x^7y^5}{xy}$.
4. $\frac{x^9y^8}{x^2y^2}$.
5. $\frac{1.04^4}{1.04^2}$.
6. $\frac{-9.6\,m^4}{-0.4\,m}$.
7. $\frac{135\,a^4c}{5\,ac}$.
8. $\frac{96\,x^4y^4}{-6\,xy}$.
9. $\frac{-119\,m^7n^5}{7\,m^5n^5}$.
10. $\frac{144\,m^4n^6}{-4\,m^4n}$.
11. $\frac{-175\,x^2y^4z^6}{-5\,x^2yz^5}$.
12. $\frac{-225\,m^2p^4}{-25\,mp}$.
13. $\frac{3^2 \cdot 5^3 \cdot 7^4}{5^2 \cdot 7^4}$.
14. $\frac{603\,a^8b^7c^6}{-9\,ab}$.
15. $\frac{-504\,x^{10}y^9z}{3\,x^9y^8}$.
16. $\frac{25\,a^mb^mc^m}{-5\,abc}$.

104. Division of a Polynomial by a Monomial. If we divide 10 ft. 8 in. by 2, we have 5 ft. 4 in. Similarly, we have

$2)\underline{10 \text{ ft. } 4 \text{ in.}}$ $\quad$ $4)\underline{40 \text{ yd. } 16 \text{ in.}}$ $\quad$ $9)\underline{9 \text{ tens} + 9}$

$5 \text{ ft. } 2 \text{ in.}$ $\quad$ $10 \text{ yd. } 4 \text{ in.}$ $\quad$ $1 \text{ ten} + 1$

$2)\underline{10f + 4i}$ $\quad$ $4)\underline{40 \cdot 3 + 16 \cdot 5}$ $\quad$ $9)\underline{9t + 9}$

$5f + 2i$ $\quad$ $10 \cdot 3 + 4 \cdot 5$ $\quad$ $t + 1$

That is, to divide a polynomial by a monomial,

Divide each term of the dividend by the divisor and add the partial quotients.

If we divide zero by any number, the result we obtain is zero. Division of any number by zero has no meaning.

Exercise 85. Division of a Polynomial by a Monomial

Examples 1 to 9, oral

1. Divide by 2: 8 ft. 6 in.; $8f + 6i$; 8 tens + 6; $8t + 6$; 86.
2. Divide by 3: 9 mi. 15 rd.; $9m + 15r$; $9xy + 15xy$.
3. Divide by a: $ax + ay$; $abc + axy$; $a\sqrt{x} + a\sqrt{y}$; $a + a^2$.

Divide the following:

4. $px^2 + py^2$ by p.
5. $axy + amn$ by a.
6. $mpq - mxy$ by m.
7. $-ax + 0$ by a.
8. $-ax + ay$ by $-a$.
9. $am^2 - an^2 + ax^2$ by a.

Perform the indicated divisions:

10. $\dfrac{35\,p^4m + 75\,p^2m^3}{5\,p^2m}$.
11. $\dfrac{64\,a^2b^3c^4 - 48\,a^4b^3c^2}{-4\,a^2b^2c^2}$.
12. $\dfrac{75\,a^5b^4c^3 - 65\,a^3b^4c^5}{-5\,a^3b^3c^3}$.
13. $\dfrac{25\,m^3p + 35\,m^4q - 15\,m^5r}{-5\,m^3}$.
14. $\dfrac{ab^2c^3 - a^3b^2c + a^2b^2c^2}{ab^2c}$.
15. $\dfrac{-4\,ab^2x - 6\,a^2by + 8\,abxy}{-2\,ab}$.

105. Division of a Polynomial by a Polynomial. The following typical case of the division of a polynomial by a polynomial should be carefully studied:

Divide $x^2 - 5x - 80$ by $x + 7$.

OPERATION

$$\begin{array}{r|l} x^2 - 5x - 80 & x + 7 \\ \underline{x^2 + 7x} & \overline{x - 12} \\ -12x - 80 & \\ \underline{-12x - 84} & \\ 4 & \end{array}$$

CHECK

Let $x = 1$. Then

$$\frac{-84 - 4}{8} = -11$$

Dividing x^2, the first term of the dividend, by x, the first term of the divisor, we have x, the first term of the quotient.

Multiplying the divisor by x, we have $x^2 + 7x$, which we subtract from the dividend, leaving $-12x - 80$ still to be divided.

Dividing $-12x$ by x, we have -12, the second term of the quotient.

Multiplying the divisor by -12, we have $-12x - 84$, which we subtract, leaving a remainder 4.

Hence the complete quotient is $x - 12$ and the remainder is 4. We often write the result in the compact form $x - 12 + \frac{4}{x - 12}$.

We may check the result (1) by carefully reviewing the work, (2) by multiplying the quotient by the divisor and adding the remainder, or (3) by substituting some convenient values for the letters.

Applying the second of these checks, we can easily show that $(x + 7)(x - 12) + 4 = x^2 - 5x - 80$.

Applying the third check, letting $x = 1$, we have $1 - 5 - 80 = -84$, $1 + 7 = 8$, $(-84 - 4) \div 8 = -11$; and the quotient, $x - 12$, becomes -11.

In the check we cannot put -7 for x, because this would make the divisor zero, but we can use any other number.

Since we cannot divide by zero, in the check for division we use for the letters values that do not make the divisor zero.

In commercial algebra we rarely need to divide by a polynomial of more than two terms. The principle is, however, the same, and a few examples are given on page 123.

Exercise 86. Division of a Polynomial by a Polynomial

Examples 1 to 5, oral

1. In dividing $x^3 + 3x^2y + 2xy^2$ by $x^2 + 2xy$, what is the first term of the quotient?

2. How do you check the work in division?

In the following examples state the first term of each quotient:

3. $x^2 - 5x + 6$ divided by $x - 3$; by $x - 2$.

4. $8m^2 - 10mn - 3n^2$ divided by $4m + n$; by $2m - 3n$.

5. $p^3 - 8p - 3$ divided by $-p + 3$; by $p^2 + 3p + 1$.

Divide the following, checking the results:

6. $a^2 + 3a + 2$ by $a + 1$.
7. $b^2 + 5b + 4$ by $b + 1$.
8. $c^2 + 8c + 12$ by $c + 2$.
9. $d^2 - 2d - 15$ by $d + 3$.
10. $e^2 + 11e + 28$ by $e + 4$.
11. $a^2 - 5ab - 6b^2$ by $a + b$.
12. $a^2 - 6ab + 5b^2$ by $a - b$.
13. $c^2 + cd - 6d^2$ by $c + 3d$.
14. $p^2 - 8pq + 22q^2$ by $p - 5q$.
15. $x^2 + xy - 22y^2$ by $x - 4y$.

Arrange according to the descending powers of x, and divide:

16. $x^3 + x^4 - 16x - 4 - 9x^2$ by $4 + x^2 + 4x$.
17. $6 + x^4 - 12x + 11x^2 - 5x^3$ by $3 - 3x + x^2$.
18. $2x^4 + 9x - 12 - 5x^3 - 7x^2$ by $1 - x + x^2$.
19. $-7x^3 - 10x^2 + x^4 - 5 + 21x$ by $x + x^2 - 3$.
20. $3x^3 + x^4 - 7x - 4x^2 + 5$ by $3x + x^2 - 2$.
21. Divide $2h + 8t + 8u$ by $h + 4t + 4u$; 288 by 144.
22. Divide $7T + 8h + 7t + 5u$ by $2h + 2t + 5u$.

As in Ex. 21, the teacher should show the relation of this work to arithmetic. Indeed, the chief object of teaching the division of polynomials in commercial algebra is to throw light upon the numerical work.

106. Short Methods with Numbers. Algebra enables us to see the reasons for many of the short methods used in commercial work. The following are some of the cases:

1. *To divide by 5, multiply by 2 and move the decimal point one place to the left.*

For $a \div 5 = \frac{a}{5} = \frac{2\,a}{10}$, so that to divide a (any number) by 5 we may simply multiply by 2 and then divide by 10. It is much easier to multiply by 2 and move the decimal point than it is to divide by 5.

2. *To divide by $12\frac{1}{2}$, which is $\frac{100}{8}$, multiply by 8 and move the decimal point two places to the left.*

3. *To divide by a number, divide by any one of its factors, divide the quotient thus found by any other of the factors, and so continue until all the factors of the divisor have been used.*

For it is evident that $abc \div ab = \frac{abc}{a} \div b = bc \div b = c$. That is, to divide by ab we may first divide by a and then divide the quotient by b.

This is often convenient, as in the case of $146.70 ÷ 45. We may first divide by 9 and then by 5, this being a little easier than to divide by 45.

In the case explained above there may be remainders. Thus, in dividing 14,699 by 45, the first quotient is 1633 and the remainder is 2 units. The second quotient is 326, and since we are dividing the number of 9's in 14,669 by 5, the remainder is 3 nines, or 27.

Hence the total remainder is $2 + 3 \times 9$, or 29.

To find the remainder in dividing by factors, multiply each partial remainder by all the divisors preceding the one which produced it. The sum of these products, together with the remainder from the first division, is the remainder sought.

$$\begin{array}{r|l} 9 & \$146.70 \\ \hline 5 & \$16.30 \\ \hline & \$3.26 \end{array}$$

$$\begin{array}{r|l} 9 & 14699 \\ \hline 5 & 1633 \quad 9\text{'s} + 2 \\ \hline & 326 \quad 45\text{'s} + 3\ 9\text{'s} \end{array}$$

Quotient $= 326$

Remainder $= 2 + 3 \times 9 = 29$

Exercise 87. Short Methods

Examples 1 to 7, oral

1. Since $a \div 25 = 4\,a \div 100$, what is a short method for dividing a number by 25? Illustrate by the case of $15{,}475 \div 25$.

State the short methods suggested by the following equations:

2. $a \div 125 = \dfrac{8\,a}{1000}$.

3. $a \div 50 = \dfrac{2\,a}{100}$.

4. $a \div 33\frac{1}{3} = \dfrac{3\,a}{100}$.

5. $a \div 66\frac{2}{3} = \dfrac{a + \frac{1}{2}\,a}{100}$.

6. $a \div 75 = \dfrac{a + \frac{1}{3}\,a}{100}$.

7. $a \div 37\frac{1}{2} = \dfrac{2\,a + \frac{2}{3}\,a}{100}$.

8. Find a short method for dividing a number by $62\frac{1}{2}$.

9. Find a short method for dividing a number by $87\frac{1}{2}$.

Perform the following divisions by the method of dividing by factors:

10. $357 \div 21$.

11. $725 \div 25$.

12. $486 \div 27$.

13. $1024 \div 32$.

14. $2135 \div 35$.

15. $4270 \div 35$.

16. $2352 \div 42$.

17. $2352 \div 49$.

18. $2352 \div 48$.

Divide the following by dividing by factors and find the remainders:

19. $721 \div 35$.

20. $642 \div 21$.

21. $1000 \div 42$.

22. $2000 \div 36$.

23. $1111 \div 25$.

24. $2002 \div 27$.

Using short processes, divide the following:

25. $\$282.36 \div 75$.

26. $\$5.63 \div \$0.12\frac{1}{2}$.

27. $\$35.75 \div \$0.62\frac{1}{2}$.

28. $\$185 \div \$0.20$.

29. At the rate of 6 lb. for \$1, how many pounds of white grapes can be bought for \$4.50?

107. Divisibility of Numbers. In making the computations needed in commercial work we frequently have to determine whether or not one number is divisible by another, by which we mean divisible without a remainder. For example, we frequently wish to know if, in a large number of articles, we have exact dozens, exact half dozens, or exact lots of 15 or 25.

You learned something in arithmetic about this work, but you knew little of the reason for it and may have forgotten much of it. Algebra makes the whole matter very clear and enables us to work out the rules for ourselves if we happen to forget them.

The following are some of the tests for divisibility:

A number is divisible by 2 if the units' figure is divisible by 2.

This is a short way of saying that it is divisible by 2 if the number represented by the units' figure is divisible by 2. Such short expressions are allowable if the meaning is clear, and will be used when convenient.

Any number is equal to its tens plus its units. For example, 1275 is equal to 127 tens + 5 units, and 7 is equal to 0 tens + 7 units. Hence any number may be represented by t tens + u units, or by $10t + u$. Now $10t$ is certainly divisible by 2, and so the number $10t + u$ is divisible by 2 if u is divisible by 2.

A number that is divisible by 2 is called an *even number*; a number that is not divisible by 2 is called an *odd number*.

A number is divisible by 5 if the units' figure is divisible by 5.

The proof, which is practically the same as the proof for divisibility by 2, is left to the student.

A number is divisible by 4 if the number represented by the last two digits at the right is divisible by 4.

Any number is equal to its hundreds plus its tens plus its units and hence may be represented by h hundreds + t tens + u units, or by $100h + 10t + u$. Now $100h$ is certainly divisible by 4, and so the number is divisible by 4 if $10t + u$ is divisible by 4; that is, if the number represented by the last two digits is divisible by 4.

A number is divisible by 8 if the number represented by the last three digits is divisible by 8.

Any number is equal to $1000\,T + 100\,h + 10\,t + u$. But $1000\,T$ is certainly divisible by 8, and so the whole number is divisible by 8 if $100\,h + 10\,t + u$ is divisible by 8.

A number is divisible by 9 or by 3 if the sum of its digits is divisible by 9 or by 3 respectively.

Any number may be represented by calling its units a, its tens b, its hundreds c, and so on, and then writing it by beginning with the units, thus: $a + 10\,b + 100\,c + 1000\,d + \cdots$, as far as we wish to go. It may also be written

$$a + 9\,b + b + 99\,c + c + 999\,d + d + \cdots$$

or $$(a + b + c + d + \cdots) + (9\,b + 99\,c + 999\,d + \cdots).$$

Now the second parenthesis is certainly divisible by either 9 or 3. Hence the number is divisible by 9 or by 3 if the first parenthesis, which is the sum of the digits, is divisible by 9 or by 3 respectively.

For example, the number 30,003 is divisible by 3 but not by 9, and the number 864 is divisible by either 9 or 3.

A number is divisible by 11 if the difference between the sums of its digits of odd order and of even order is divisible by 11.

Thus 207,988 is divisible by 11. We have 8 + 9 + 0, or 17, the sum of the digits of odd order (that is, in the 1st, 3d, and 5th places) and 8 + 7 + 2, or 17, the sum of the digits of even order. Now $17 - 17 = 0$, and 0 is divisible by 11, the quotient being 0.

As in the case of 9, we may represent any number by

$$a + 10\,b + 100\,c + 1000\,d + 10{,}000\,e + \cdots,$$

or by $$a + 11\,b - b + 99\,c + c + 1001\,d - d + 9999\,e + e + \cdots,$$

or by $$(a - b + c - d + e - \cdots) + (11\,b + 99\,c + 1001\,d + 9999\,e + \cdots).$$

The second parenthesis is certainly divisible by 11. Hence the number is divisible by 11 if the first parenthesis, the difference between the sums of the digits of odd order and even order, is divisible by 11.

A number is divisible by 6 if it is even and if the sum of the digits is divisible by 3.

For we need simply to apply the tests for divisibility by 2 and by 3.

Exercise 88. Divisibility

1. State and prove the test for divisibility by 10.

2. Prove that a number is divisible by 12 if the sum of the digits is divisible by 3 and if the number represented by the last two digits is divisible by 4.

3. Prove that a number is divisible by 7 if the sum of the odd periods minus that of the even periods is so divisible.

Let x, y, z represent the third, second, and first periods respectively of a number of three periods. Then the number will be represented by $1{,}000{,}000\,x + 1000\,y + z$, or by $999{,}999\,x + x + 1001\,y - y + z$, or by $(999{,}999\,x + 1001\,y) + (x - y + z)$. But the first parenthesis is certainly divisible by 7, and hence the whole number is divisible by 7 if $x - y + z$, or $x + z - y$, is divisible by 7. But $x + z - y$ is the sum of the odd periods minus the even period.

The proof is similar to this if more than three periods are taken. The student should write the proof for the case of four periods.

Find all possible divisors from 2 to 12 of the following:

4. 1158.	**7.** 1337.	**10.** 1810.	**13.** 9033.	**16.** 10,002.
5. 1055.	**8.** 2583.	**11.** 3091.	**14.** 2809.	**17.** 10,010.
6. 4809.	**9.** 1384.	**12.** 5001.	**15.** 7007.	**18.** 18,402.

Without written division, write those of the following numbers that are divisible by 7:

19. 7014.	**21.** 13,021.	**23.** 701,707.	**25.** 201,714.
20. 14,721.	**22.** 700,707.	**24.** 700,714.	**26.** 722,001.

Write such of the following numbers as are divisible by 9 and such as are divisible by 11, together with the work showing the short process used:

27. 69,876.	**29.** 88,088.	**31.** 37,466.	**33.** 54,045.
28. 76,747.	**30.** 42,020.	**32.** 80,280.	**34.** 54,560.

Exercise 89. Review of Chapter VII

1. Divide 800×1.06^5 by 1.06^3; $300 \times 1.04^5 + 200 \times 1.04^4$ by 1.04^2.

2. Find the value of $\dfrac{30 \times 7^3 + 5 \times 13^4 \times 7^2 - 5 \times 6^2 \times 0}{5 \times 7^2}$.

3. Divide $64\,x^7y - 72\,x^5y^2 + 40\,x^3y^3 - 32\,xy^4$ by $-8\,xy$.

4. Divide $9 - x^3 - 27\,x^2 + 12\,x^4 + 3\,x$ by $-3 + 4\,x^2 - 3\,x$, and check the result.

5. Derive a short method for dividing any number a by $62\frac{1}{2}$; by $37\frac{1}{2}$; by $16\frac{2}{3}$; by $14\frac{2}{7}$.

6. Resolve into factors, using the principles of divisibility of numbers: 555, 609, 1449, 489, 7014.

7. How many tubs, holding 75 lb. each, will be needed to ship a consignment of 9600 lb. of butter? What short method should you employ in solving this example?

8. Write two exact divisors of 4899, using the principle that $a^2 - b^2$ is exactly divisible by $a + b$ and by $a - b$.

9. Divide $20\,a^2 + 18\,am - 18\,m^2$ by $5\,a - 3\,m$, and check the result.

10. Divide \$7854.85 by 1.0612, and check the result by the check of 9's.

11. Divide $a^3 - 3\,a^2b + 3\,ab^2 - b^3$ by $a - b$, and check the result.

12. Divide $x^4 - 81$ by $x - 3$, and check the result.

13. Divide $a^5 - b^5$ by $a - b$, and check the result.

14. At $37\frac{1}{2}¢$ a yard, how many yards of madras can a girl buy for \$4.50?

Use short methods in solving Exs. 14 and 15.

15. If a keg of nails weighs $62\frac{1}{2}$ lb., how many kegs can safely be stored on a platform built to carry 2 T.?

Exercise 90. Review of Chapters I–VII

1. A trapezoid has bases whose lengths are m and n and whose altitude is k. What is the area?

2. What is the value of $\sqrt{3\,ab^2c^4}$ if $a=3$, $b=2$, $c=5$?

3. Solve the equations $3x+7=13$; $\frac{1}{3}x+2=5$; $x-2=9$.

4. A mixture of 18 parts, by weight, of alcohol and 11 parts of water is to be prepared, the total to be 15 oz. How much alcohol and how much water must be used?

5. Find the average temperature for a day if the records every 3 hr. are $+8°$, $+11°$, $+12°$, $+5°$, $0°$, $-3°$, $-5°$, $-9°$.

6. From the equation $a-b+b=a$, write a rule for testing the result of subtraction in arithmetic.

7. By how much does $p(1+rt)$ exceed p?

8. By how much does a^2-b^2 exceed $2\,b^2$?

9. What number must be added to a to produce b?

10. What number must be added to $a-b$ to produce $2\,a-3\,b$?

11. If a dealer's profit on a certain grade of shoes is \$1.80 per pair, what is his profit on 350 pairs?

12. If a dealer's profit on a pair of shoes is p dollars, what is his profit on n pairs?

13. If the cost of a pair of shoes is c dollars, the profit desired is p dollars per pair, and the number of pairs sold is n, write the equation expressing the total receipts, r dollars, from the sale of the n pairs of shoes.

14. Multiply $3x^2+17x+10$ by $2x-7$, and then divide the result by $3x+2$.

15. The selling price is s dollars, cost c dollars, overhead o dollars, and profit p dollars. Write the formula for p.

CHAPTER VIII

SIMPLE EQUATIONS

108. Simple Equation. An equation which contains the first power and no higher power of the unknown quantity is called a *simple equation.*

Such an equation is also said to be of the *first degree.* It is also called, particularly in higher mathematics, a *linear equation.* We have already considered in Chapter II the easier kinds of simple equations.

109. Identity. An equation which is true for any value whatsoever of any letter or letters is called an *identity.*

Thus $(a + b)^2 = a^2 + 2ab + b^2$ is an identity, for it is true if $a = 3$, $b = 4$, or if the letters have any other values whatsoever. An identity is sometimes expressed by the symbol $\equiv$, as in $a + b \equiv b + a$, read "$a + b$ is identical to $b + a$." The term is also used to represent such numerical equalities as $2 + 3 = 5$.

110. Satisfying an Equation. The number which, substituted for the unknown quantity, reduces an equation to an identity is said to *satisfy* the equation.

This number is the value of the unknown quantity. Thus, if we have $x + 7 = 9$, then $x = 2$, and 2 satisfies the equation.

111. Solving an Equation. To find the value or values of the unknown quantity which will satisfy an equation is to *solve* the equation.

Thus, if $x + 3 = 5$, then $x = 2$; for if we put 2 for x, the two members of the equation become $2 + 3$ and 5, which are the same in value. In other words, the equation then becomes an identity.

Similarly, if $x + a = 3a$, then $x = 2a$; for if we put $2a$ for x, the two members become $2a + a$ and $3a$, which are the same in value.

Many equations can be solved by simple inspection.

112. Root. A value of the unknown quantity which satisfies an equation is called a *root* of the equation.

Thus, in $x - 4 = 9$, the root of the equation is 13.

A simple equation has only one root. There are, however, equations that have more than one root. For example, the equation $x^2 - 3x + 2 = 0$ has two roots because it is satisfied if $x = 1$ and also if $x = 2$.

113. Transposition. We have learned that if we subtract equals from equals, we have equals left. We have also seen that this amounts to taking a number from one side of the equation, changing the sign, and putting it on the other side.

Taking a number from one side of an equation, changing its sign, and putting it on the other side is called *transposition.*

For example, if $2x - 4 = 8 + x$,

we may subtract x from each side, or transpose x, and have

$$x - 4 = 8.$$

We may then subtract -4 from each side (or add $+4$ to each side) and have

$$x = 12.$$

The word "transpose," while sometimes convenient, is evidently not necessary, since we can use "subtract" or "add" in its stead.

114. Numerical Equation. An equation containing no letters except those representing roots is called a *numerical equation.*

Thus $2x + 3 = 5$ is a numerical equation, but $ax + b = 4$ is not.

Solve the equation $4x - 7 + x = 10 - 3x - 1$.

Combining like terms, $5x - 7 = 9 - 3x$.

Subtracting -7, $5x = 16 - 3x$.

Subtracting $-3x$, $8x = 16$.

Dividing by 8, $x = 2$.

Check. $4 \times 2 - 7 + 2 = 10 - 3 \times 2 - 1 = 3$.

The solution may be shortened, thus:

Combining like terms, $5x - 7 = 9 - 3x$.

Adding 7 and $3x$, $8x = 16$.

Dividing by 8, $x = 2$.

Exercise 91. Numerical Equations

Examples 1 to 9, oral

1. Is $x + 3 = 4$ an identity? State the reason.

2. Is $x + 3 = 3 + x$ an identity? State the reason.

3. What value of x will satisfy the equation $x - 4 = 5$? Prove it.

4. What is the root of the equation $x + 7 = 12$? Prove it.

5. What right have you to transpose -3 in the equation $x - 3 = 9$? Explain why you must change the sign.

Solve the following equations:

6. $x - 4 = 6$. **7.** $x + 5 = 9$. **8.** $x + 7 = 15$.

9. In solving the equation $4x + 5 - 2x = 15$ what is the first step? the second step? the third step?

Solve the following equations:

10. $5x - 7 + 3x = 9$. **12.** $8 + 2x - 19 = x$.

11. $7x - 4 - 5x = 10$. **13.** $5 - x - 8 + 6x = 17$.

14. If to 5 times a certain number we add 2, the sum is 20 diminished by 4 times the number. What is the number?

Solve the following equations:

15. $5x - 7 + 3x + 6 + 8x = 13 + 6x + 40$.

16. $7x - 2 + 9x - 3 + 5x = 4 + x + 31$.

17. $8x + 7 - 2x + 5 = 4x + 12 - (x - 30)$.

18. If 4 is increased by three times a certain number, the result is 19. Find the number.

19. Four times a certain number is diminished by 3, the result being 29. Find the number.

115. Two Unknowns, One Letter. A problem may involve the finding of the values of two unknown quantities and still require the use of only a single letter if there are two statements either expressed or implied.

As will be seen, one statement will give a relation which makes it possible to represent the two unknowns by the aid of only a single letter, but with different coefficients; the other statement will be one which can be translated into an equation.

1. On June 21 daylight in a certain place lasts 8 hr. longer than darkness. How many hours are daylight?

Statement I. Daylight = darkness + 8 hr.
Statement II (implied). Daylight + darkness = 24 hr.

Let x = the number of hours of darkness.
Then, from I, $x + 8$ = the number of hours of daylight.

But because, from Statement II,

Daylight + darkness = 24 hr.,

we have $x + 8 + x = 24$,

or $2x + 8 = 24$.

Solving, $x = 8$, the number of hours of darkness.
Hence $x + 8 = 16$, the number of hours of daylight.

2. A clerk counting change consisting of dimes and quarters remembers that there were 45 coins and that their total value was \$8.40. How many were there of each?

Statement I. Number of quarters + number of dimes = 45.
Statement II. Value of quarters + value of dimes = \$8.40.

Let x = the number of quarters.
Then, from Statement I,

$45 - x$ = the number of dimes.

Multiplying 25 by the number of quarters and 10 by the number of dimes, to find the values in cents, we have

$$25x + 10(45 - x) = 840.$$

Solving, $x = 26$, the number of quarters.
Hence $45 - x = 19$, the number of dimes.

Exercise 92. Problems

Examples 1 to 9, oral

1. Five more than three times a certain number is 20. What is the number?

2. The sum of two numbers is 34 and one of the numbers is 19. What is the other number?

3. The difference between two numbers is 17 and the larger number is 32. What is the smaller number?

4. The difference between two numbers is 41 and the smaller number is 19. What is the larger number?

5. The product of two numbers is 51 and one of the numbers is 17. What is the other number?

6. The quotient of two numbers is 27 and the divisor is 3. What is the dividend?

7. The quotient of two numbers is 32 and the dividend is 96. What is the divisor?

8. The difference between two numbers is 12 and one of the numbers is 20. What is the other number?

Notice that there are two possible answers to Ex. 8, and also to Ex. 9.

9. The quotient of two numbers is 10 and one of the numbers is 20. What is the other number?

10. The sum of two numbers is 32 and one of the numbers is three times the other. Find the numbers.

11. The sum of two numbers is 36 and one of the numbers is 2 more than the other. Find the numbers.

12. The sum of two numbers is 36 and one of the numbers is 2 less than the other. Find the numbers.

13. The sum of two numbers is 5.6 and one of the numbers is 4.2 more than the other. Find the numbers.

14. Two partners, A and B, have an agreement to allow A to receive from the profits \$1200 as rent for the building which he owns and which they occupy, and then to divide the remainder of the profits so that A shall receive twice as much of this remainder as B. If the profits for the year, less all overhead charges excepting rent, are \$7500, how much does each receive, including the rent in A's share?

15. If A owns 9 shares of a business and B owns the remaining 11 shares, and if they divide profits accordingly, how shall they divide a profit of \$6000?

16. In a commercial transaction two men made \$4500. By their articles of agreement the second was to receive 50% more than the first. How much did each receive?

17. A merchant put in an order for a lot of goods and a week later gave an order for $12\frac{1}{2}\%$ as much. He paid \$2700 for all the goods. How much was each order?

18. A grocer makes a blend of coffee by using 2 parts of Mocha and 5 parts of Rio. How many pounds of each must he take to make 100 lb. of the blend?

19. A fireman's experience has shown that 1 T. of anthracite coal fills 34 cu. ft. of space, and 1 T. of bituminous fills 42 cu. ft. He has 3200 cu. ft. of space available for storage, and needs three times as many tons of bituminous coal as of anthracite. How many tons of each should he order?

20. A mountain railroad, X, has a freight tariff per mile $1\frac{1}{2}$ times that of another road, Y. Some freight "billed through" for \$87.50 was hauled 85 miles by X and 120 miles by Y. How should they divide the \$87.50?

21. A invests \$5000 more capital than B in a business. After realizing a profit of 20% on the investment, their free assets are \$75,000. What is each man's share of the assets?

116. Problems relating to Mixtures. In various kinds of business it becomes necessary to mix certain ingredients, and the quantities are often found easily by algebra.

How much water must be added to 12 qt. of a 25% solution of alcohol to reduce it to a 10% solution?

Let x = the number of quarts to be added.

Then $12 + x$ = the total number of quarts,

and $10\%(12 + x)$ = the number of quarts of alcohol.

But 25% of 12 = the number of quarts of alcohol.

Therefore $10\%(12 + x) = 25\%$ of 12.

Combining, $1.2 + 10\%\, x = 3.$

Subtracting 1.2, $10\%\, x = 1.8$

Dividing by 10%, $x = 18.$

Therefore 18 qt. of water must be added.

Check. 25% of 12 qt. = 3 qt., the amount of alcohol.

12 qt. + 18 qt. = 30 qt., the amount of the mixture.

Furthermore, 3 qt. is 10% of 30 qt., as required.

Exercise 93. Problems relating to Mixtures

1. How much water must be added to a gallon of a 22% solution of alcohol to reduce it to an 11% solution?

2. How much water must be added to 63 gal. of alcohol 95% pure to reduce it to a $66\frac{2}{3}\%$ solution?

3. How many pounds of water must be added to 320 lb. of a 4% solution of salt to reduce it to a $2\frac{1}{2}\%$ solution? How many gallons, allowing 8.35 lb. of water to a gallon?

4. How much water must be added to a pint of a 10% solution of a certain medicine to reduce it to a 2% solution?

5. How much water must be added to a barrel of vinegar 80% pure to reduce it to a 60% solution?

117. Problems relating to Motion. Algebra is often useful in solving problems relating to motion.

1. A train leaves Pittsburgh for the West at the same time that another train leaves Pittsburgh for the East. The former travels at the average rate of 42 mi. an hour and the latter at the average rate of 38 mi. an hour. In how many hours will the trains be 240 mi. apart?

Let x = the number of hours required.

In 1 hr. the trains are $(42 + 38)$ mi. apart, or 80 mi. apart.
In x hours the trains are $x \cdot 80$ miles apart, or $80x$ miles.

Therefore $80x = 240.$

Dividing by 80, $x = 3.$

Therefore the trains will be 240 mi. apart in 3 hr.

Check. $3 \times (42 + 38)$ mi. $= 240$ mi.

2. A man starts from a certain place and rides on his bicycle at the rate of 16 mi. an hour. Forty-five minutes later an automobile starts after him from the same place and travels at the rate of 24 mi. an hour. How long will it take the automobile to overtake him?

In 45 min. the bicycle has gone $\frac{3}{4}$ of 16 mi., or 12 mi.

Let x = the number of hours required.

Since the automobile gains in an hour 24 mi. $-$ 16 mi., or 8 mi.,

$8x$ = the number of miles gained in x hours.

But 12 = the number of miles to be gained.

Therefore $8x = 12.$

Dividing by 8, $x = 1\frac{1}{2}.$

Therefore the automobile will overtake the bicycle in $1\frac{1}{2}$ hr.

Check. In $1\frac{1}{2}$ hr. the automobile will travel $1\frac{1}{2} \times 24$ mi., or 36 mi., and the bicycle $1\frac{1}{2} \times 16$ mi., or 24 mi., and 24 mi. $+$ 12 mi. $=$ 36 mi.

In all problems relating to motion the average rate is to be understood unless the contrary is stated.

Exercise 94. Problems relating to Motion

Examples 1 to 4, oral

1. At 3 mi. an hour, how far will a man walk in 4 hr.? in $\frac{1}{2}$ hr.? in t hours? in $(a+b)$ hours?

2. At r miles an hour, how far will an airplane go in t hours? in $2t$ hours? in $\frac{1}{2}$ hr.?

3. If two men travel from the same place, one going east at the rate of 3 mi. an hour and the other west at the rate of 40 mi. an hour, how far apart are they in 2 hr.? in h hours?

4. If two men start from Chicago and New York at the same time and travel toward each other, the first at the rate of a miles an hour and the second at the rate of b miles an hour, how much nearer each other will they be in t hours?

5. Two men start from Denver at the same time, one traveling south 35 mi. an hour, and the other north 38 mi. an hour. How many miles apart will they be in 3 hr.? In how many hours will they be 292 mi. apart?

6. Two men start from the same place, one going east and the other going west, the former traveling twice as fast as the latter. In 4 hr. they are 300 mi. apart. Find the rate of each.

7. Two men start from the same place, one to the north and the other to the south, the former traveling 5 mi. an hour faster than the latter. In 2 hr. they are 150 mi. apart. Find the rate of each.

8. A man leaves a friend at the railway station and starts to drive north just as his friend leaves on the train for the south. A half hour later they are 20 mi. apart. If the man drives one fourth as fast as the train travels, what is the rate of each?

118. Three Unknowns, One Letter. **1.** Of the 50 shares of a business A owns 13 shares, B owns 17, and C owns 20. If they divide on the basis of the number of shares owned by each, how should they divide a net profit of $25,000?

Statement I. A's share is $\frac{13}{20}$ of C's.

Statement II. B's share is $\frac{17}{20}$ of C's.

Statement III. The sum of the shares is $25,000.

Since each share receives the same profit, let x = the number of dollars profit on one share. Then, referring to the number of dollars in each partner's share of the total profit,

$20\,x$ = C's share, $17\,x$ = B's share, and $13\,x$ = A's share.

Hence $13\,x + 17\,x + 20\,x = 25{,}000.$

Solving, $x = 500.$

Therefore $13\,x = 6500$, $17\,x = 8500$, and $20\,x = 10{,}000.$

That is, A's share is $6500, B's $8500, and C's $10,000.

2. A dealer has charged $10\,¢$ a peck more for first grade potatoes than for second grade and $15\,¢$ a peck more for second grade than for third grade. After sorting a consignment of 10 bu. that cost him $15, he finds that he has 5 bu., 3 bu., and 2 bu. respectively of the three grades. To maintain the above differentials and to make a profit of $5 on the consignment, how much must he charge for a peck of each grade?

Statement I. Price per peck of second grade = price per peck of third grade + $15\,¢$.

Statement II. Price per peck of first grade = price per peck of second grade + $10\,¢$.

Statement III. Total receipts = $20.

Let x = the number of cents per peck of third grade.

Then $x + 15$ = the number of cents per peck of second grade,

and $x + 25$ = the number of cents per peck of first grade.

Hence $20\,(x + 25) + 12\,(x + 15) + 8\,x = 2000.$

Solving, $x = 33.$

Therefore $x + 15 = 48$, and $x + 25 = 58.$

Hence the dealer must charge $58\,¢$, $48\,¢$, and $33\,¢$ respectively.

Exercise 95. Three Unknowns, One Letter

1. Three partners, A, B, and C, own respectively 20, 25, and 55 shares of the 100 shares of stock of their business. A owns the building and is allowed a yearly rental of \$2400. After paying all overhead charges except the rent, they have \$14,400. How should this \$14,400 be divided among them?

2. A fruit dealer has charged 25¢ per bushel more for apples of grade A than for those of grade B, and 20¢ more for those of grade B than for those of grade C. After sorting 90 bu. that cost him \$150, he finds that he has 20 bu., 30 bu., and 40 bu. respectively of the three grades. If he wishes to maintain the above differentials and realize a profit of \$30, how much must he charge per bushel for each grade?

3. A taster has estimated a blend of tea as worth 80¢ per pound. The blend is made of 3 parts of a 50¢ tea, 2 parts of a 60¢ tea, and 1 part of a \$1 tea. How many pounds of each kind must be taken to make 30 lb. of the same blend, and what profit will be realized on its sale?

4. An insurance company has found it expedient to adopt this rule in investing its funds: to place 60% of its funds in long-time securities, 15% in mortgages, and the remaining 25% in loans to policy holders. On a certain date the company receives \$75,000, which it desires to invest. What amount shall it put into each kind of investment?

5. An experience table shows that approximately $\frac{3}{4}\%$ of persons reaching the age of 21 years die during the following year; of those reaching the age of 22 years, $\frac{3}{4}\%$ die during the year; and of those reaching the age of 23 years, $\frac{3}{4}\%$ die during the year. If a group of persons insured at the age of 21 number 194,000 at the close of the age 23, how many were probably in the original group at the age of 21?

119. Tabular Arrangement. 1. A rectangular assembly hall is 10 ft. longer than wide, and if each dimension is increased 10 ft. the area is increased 800 sq. ft. Find the dimensions of the hall and the area of the floor.

Given Conditions	Length	Width	Area
First condition . .	$x+10$	x	x^2+10x
Second condition . .	$x+20$	$x+10$	$x^2+30x+200$
Third condition . .	$x+20$	$x+10$	$x^2+10x+800$

Such a tabular arrangement is often helpful.

Since $x^2 + 30x + 200 = x^2 + 10x + 800,$

we have $x = 30;$

whence $x + 10 = 40,$

and $x^2 + 10x = x(x+10) = 30 \times 40 = 1200.$

That is, the hall is 30 ft. wide, 40 ft. long, and 1200 sq. ft. in floor area.

2. A sum of money invested at 4% yields an income \$300 larger than half the sum yields at 5%. Find the amount invested at each rate and the income from the first amount.

Given Conditions	Sum	Rate	Income
First condition . .	$2x$	4%	$0.08x$
Second condition . .	x	5%	$0.05x$
Third condition . .	$2x$	4%	$0.05x+300$

Since $0.08x = 0.05x + 300,$

we have $0.03x = 300.$

Dividing by 0.03, $x = 10{,}000;$

whence $2x = 20{,}000.$

That is, the investment at 5% is \$10,000, and the investment at 4% is \$20,000, the income from the latter being \$800.

Exercise 96. Tabular Arrangement

Using tabular arrangement, solve the following problems:

1. The cost of some silk at \$2.20 per yard is the same as the cost of 20 yd. more of wool at \$1.76 per yard. Find the number of yards of each and the total cost of the goods.

2. A workman receives an offer of a position paying an increase of 80¢ per day over his present wages. He calculates that in 30 da. in the new position he will receive as much as he now earns in 40 da. Find his present rate of wages per day and his rate if he accepts the offer.

3. A man invests \$9000, part at 4% and part at 5%. If he gets an equal income from the two investments, how much does he invest in each, and what is the income from each?

4. A rectangle is 5 yd. longer than wide. If the length is increased 10 yd., and the width is increased 5 yd., the area is increased 450 sq. yd. Find the length, width, and area before the increase is made.

5. A man invested a certain sum of money in United States Liberty 4% bonds, buying them at par. If he had invested \$1500 more, and if the rate were 1% larger, he would be receiving in 1 yr. the same interest he now receives in 2 yr. How much did he invest in the bonds?

6. During the month of January, 1918, the total receipts at 16¢ a yard from the sale of galatea in a dry goods store amounted to half as much as the receipts from the same source in Jan., 1919, at 20¢ a yard. The number of yards sold in Jan., 1919, was 3000 more than in Jan., 1918. How many yards were sold in Jan., 1918, and in Jan., 1919?

7. A certain principal invested at 4% and a second principal twice as large, invested at $4\frac{1}{2}$%, together bring in \$260 interest in 1 yr. Find the amount of each principal.

120. Miscellaneous Problems. Before attempting the next exercise the following solutions should be studied.

1. The sum of three consecutive numbers is 48. Find the numbers.

Let n = the middle number.

Then $n-1$ = the smallest number,

$n+1$ = the largest number,

and $(n-1)+n+(n+1)$ = the sum.

But 48 = the sum.

Therefore $(n-1)+n+(n+1)=48$;

whence $3n=48$.

Dividing by 3, $n=16$.

Therefore the numbers are 15, 16, 17.

Check. $15+16+17=48$.

The solution is simpler if we take $n-1$, n, and $n+1$ for the numbers, instead of n, $n+1$, and $n+2$, although the latter plan is also correct.

2. A man is now three times as old as his son. Five years ago he was four times as old as his son. Find the age of each.

Let x = the number of years in the son's age.

Then $3x$ = the number of years in the father's age.

Also $x-5$ = the number of years in the son's age 5 yr. ago,

and $3x-5$ = the number of years in the father's age 5 yr. ago.

But 5 yr. ago the father was four times as old as the son.

Therefore $3x-5=4(x-5)$.

Multiplying, $3x-5=4x-20$.

Subtracting -5 and $4x$, $-x=-15$.

Dividing by -1, $x=15$.

Therefore $3x=45$.

Hence the man is 45 yr. old and his son is 15 yr. old.

Check. $45=3\times 15$. Also, 5 yr. ago they were 40 yr. and 10 yr. of age respectively, and $40=4\times 10$.

Exercise 97. Miscellaneous Problems

Examples 1 to 6, oral

1. Solve the equations $5x+7=42$; $5x-7=43$.

2. Solve the equations $2x=7$; $2x-1=10$; $2x+1=10$.

3. What number is one more than half itself?

4. A rectangle 5 in. wide has an area of 40 sq. in. What is its length?

5. A rectangular box 5 ft. long and 4 ft. wide has a capacity of 40 cu. ft. What is its depth?

6. A rectangular tank 5 ft. deep and 8 ft. wide has a capacity of 480 cu. ft. What is its length?

7. If it takes a steamer $5n$ days to go 3355 mi., what is its average rate per day?

8. After playing 20 games a baseball team had won three times as many games as it had lost. How many games had it lost? How many had it won?

9. A freight train running 35 mi. an hour leaves a station 3 hr. ahead of an express train that makes 50 mi. an hour. How soon will the express train overtake the freight?

10. The three angles of a triangle are together equal to 180°. In a certain right triangle one acute angle is three times as large as the other. Find the number of degrees in each angle.

11. The ground floor of a school building is in the form of a rectangle 60 ft. long and 49 ft. wide. An addition 35 ft. wide is to be made to cover an area equal to half the area of the original building. How long will the addition be?

12. If we can send a ten-word telegram to a certain place for 35 ¢, and a 24-word telegram for 63 ¢, what is the charge for each additional word above ten?

13. A miner succeeded in obtaining 675 oz. of silver from some ore, this being $\frac{3}{4}\%$ of the weight of the ore. How much did the ore weigh?

14. The top of a tree 120 ft. high is broken off. The length of the part broken off is four times the length of the part left standing. Find the length of each part.

15. Three men buy a summer cottage together for \$3000. The second pays twice as much as the first, and the third pays as much as the first two together. How much does each pay? One summer they plan to use the cottage for 3 mo., only one family occupying it at a time. How much of the time should each family be allowed to use it?

16. In 30 yr. from now a boy will be three times as old as he is at present. How old is he now?

17. Ten years ago a boy was 2 yr. more than one third as old as he is at present. How old is he now?

18. Four years ago a man was seven times as old as his son, and his son is now 8 yr. old. Find the age of the father at the present time.

19. A man has \$9 in half dollars and quarters, having four times as many quarters as half dollars. How many coins of each kind has he?

20. A man has \$3.85 in quarters and dimes, having three times as many dimes as quarters. How many coins of each kind has he?

21. The sum of the ages of a father and his son is 60 yr., and the father's age is 3 yr. more than twice the age of his son. Find the age of each.

22. Some boys are laying out a circular running track that is to be $\frac{1}{4}$ mi. in length. What radius must they use in describing the circle?

Exercise 98. Review of Chapter VIII

Solve the following equations:

1. $8x + 30 = 5x + 60$.

2. $9y + 1 = 5 - 3y$.

3. $(x+1)(x-1) + 6 = x^2 + x$.

4. $(x+3)^2 - (x-1)^2 - 10 = 6x$.

5. A grocer buys 6 bbl. of apples at \$3.50 a barrel, 2 bu. to the barrel. If 3 bu. spoil, at what price per bushel must he sell the remainder to realize a profit of \$5 on his investment?

6. Two stations are 90 mi. apart. A train leaves the first station traveling 5 mi. per hour faster than a train which leaves the second station at the same time. If the trains travel toward each other and are $22\frac{1}{2}$ mi. apart at the end of $1\frac{1}{2}$ hr., find the rate of each train.

7. Two stations are m miles apart. A train leaves the first station traveling e miles per hour faster than a train which leaves the second station at the same time. If the trains travel toward each other and are d miles apart at the end of h hours, find the rate of each train.

8. Substitute in the formula of Ex. 7 the values given in Ex. 6, and compare results with those found in Ex. 6.

9. How many ounces of water must be added to k ounces of an $r_1\%$ solution of alcohol to reduce it to an $r_2\%$ solution?

10. Using the result of Ex. 9 as a formula, find the number of ounces of water that must be added to 11 oz. of a 24% solution of alcohol to reduce it to a 10% solution.

11. A dealer estimates that to make his usual percentage of profit he must receive \$33 for 90 lb. of chuck, 30 lb. of sirloin steak, and 20 lb. of porterhouse steak. He wishes to charge $2\,¢$ per pound more for the porterhouse than for the sirloin steak, and $5\,¢$ per pound less for the chuck than for the sirloin. What price should he charge for each?

Exercise 99. Review of Chapters I–VIII

1. The observed temperatures for a day, taken at equal intervals of time, being $+5°$, $+9°$, $+11°$, $+12°$, $+6°$, $+3°$, $0°$, $-2°$, $-4°$, $-5°$, $-6°$, $-7°$, find the average temperature.

2. Using the formula $C = \frac{5}{9}(F - 32)$, find in centigrade degrees the average temperature found in Ex. 1.

3. Subtract $p(1+r)^n$ from $p'(1+r)^n$.

4. Subtract $500\,(1.05)^4$ from $1300\,(1.05)^4$ and find the value of the result.

5. Multiply $a+b$ by $a+c$.

6. By the method of Ex. 5 multiply 91 by 92.

7. Two men start at the same time from cities which are 50 mi. apart, and travel away from each other, one traveling 5 mi. an hour faster than the other. At the end of 3 hr. they are 245 mi. apart. Find the rate of travel of each.

8. Two men start at the same time from cities which are d miles apart, and travel away from each other, one traveling k miles an hour faster than the other. At the end of t hours they are e miles apart. Find the rate of travel of each.

9. Divide $6h^2 + 7ht + 2t^2$ by $3h + 2t$.

10. At what price must I mark goods which cost me \$800, so that I may discount the marked price 20%, 10% and still realize a profit of 20% on the cost?

11. At what price must I catalogue goods which cost \$80, so that I may allow a discount series of 20, 10, and still realize a profit of 15% on the selling price?

12. Derive a formula for operations like that of Ex. 11 by using c for cost, $d\%$ for the equivalent discount rate of the series, and $p\%$ for the profit rate.

CHAPTER IX

FRACTIONS

121. Algebraic Fraction. An expression in the form $\frac{a}{b}$, in which either a or b is an algebraic expression or both a and b are algebraic expressions, is called an *algebraic fraction.*

In commercial work we frequently need to use algebraic fractions in formulas. Instead of the above form we often use the form a/b. Each form is commonly read "a over b."

For example, $2x/y$, $\frac{2}{x}$, $\frac{x+y}{x-y}$, and $\frac{x+y}{2}$ are algebraic fractions.

Since we cannot divide by zero, the denominator cannot be zero.

122. Terms of a Fraction. In the fraction $\frac{a}{b}$, a is called the *numerator*, b is called the *denominator*, and the two together are called the *terms* of the fraction.

The numerator represents the dividend and the denominator represents the divisor of an indicated division, just as in arithmetic.

123. Reduction of Fractions. We reduce algebraic fractions to fractions having lower terms or to fractions having higher terms just as in arithmetic.

Just as $\frac{10}{15}=\frac{2}{3}$, so $\frac{a+b}{a^2-b^2}=\frac{a+b}{(a+b)(a-b)}=\frac{1}{a-b}$.

Similarly, $\frac{a}{b}=\frac{ax}{bx}$, and $\frac{a}{x}=\frac{a\cdot ax}{x\cdot ax}=\frac{a^2x}{ax^2}$.

Multiplying or dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.

Exercise 100. Reduction to Lowest Terms

Examples 1 to 3, oral

1. Reduce to lowest terms: $\frac{3}{12}, \frac{4}{16}, \frac{a}{ab}, \frac{a^3}{a^2b}, \frac{a^2}{a^3b}, \frac{a^2b^2}{a^3b}$.

2. Reduce to lowest terms: $\frac{5}{15}, \frac{ax}{ay}, \frac{a^2x}{ay^2}, \frac{2\,a^2x}{4\,ay^2}, \frac{5\,a^2x^2}{10\,a^3y^3}$.

3. Reduce to lowest terms: $\frac{abc}{a^2bc}, \frac{abc}{a^2b^2c^2}, \frac{abc^2}{a^2b^2c^2}, \frac{a^2b^2c^2}{a^3b^3c^3}$.

Reduce the following to lowest terms:

4. $\frac{8\,a^2b^3c^4}{12\,a^4b^3c^2}$. **6.** $\frac{16\cdot 5^3}{4\cdot 5^4}$. **8.** $\frac{35\,x^4y^4z}{49\,x^5y^6z}$. **10.** $\frac{385}{455}$.

5. $\frac{15\,a^4b^6c^8}{20\,a^3b^5c^{10}}$. **7.** $\frac{36\,a^{10}b^{10}}{48\,a^{12}b^{15}}$. **9.** $\frac{50\,x^8y^9z^2}{75\,x^{10}y^{10}z}$. **11.** $\frac{63\,x^ny^2}{81\,x^ny^{10}}$.

12. Reduce $\frac{(a+b)^2}{a^2-b^2}$ to lowest terms, and check the result by letting $a=3$, $b=2$.

Reduce the following to lowest terms:

13. $\frac{a^2-b^2}{(a-b)^2}$. **15.** $\frac{a^2+a}{a^2-1}$. **17.** $\frac{p^2+q^2}{p^4-q^4}$. **19.** $\frac{x^2-y^2}{x^4-y^4}$.

14. $\frac{4\,a^2-b^2}{2\,a-b}$. **16.** $\frac{m^4-n^4}{m^2+n^2}$. **18.** $\frac{x^4-16}{x^2-4}$. **20.** $\frac{9\,x^2-y^2}{3\,x-y}$.

21. Reduce $\frac{(1+2\,a)^2}{1-4\,a^2}$ to lowest terms, and check the result by letting $a=2$.

Reduce the following to lowest terms:

22. $\frac{a^2+ab}{(a+b)^2}$. **23.** $\frac{a^2-ab}{a^3-ab^2}$. **24.** $\frac{ab-b^2}{(a-b)^2}$.

124. Sign of a Fraction. The plus sign or the minus sign before a fraction is called the *sign of the fraction.*

If there is no sign expressed, the plus sign is understood as usual.

125. Changing Signs in the Terms. Since, from the law of signs in division,

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b},$$

The value of a fraction is not altered by changing the signs of both the numerator and denominator; by changing the signs of both the fraction and numerator; or by changing the signs of both the fraction and denominator.

To change the sign of the numerator means that we must change the sign of *every term* of the numerator, and similarly for the denominator. Failure to do this is the cause of many of the errors that arise.

126. Common Denominator. Two or more fractions that have the same denominator are said to have a *common denominator.*

For example, $\frac{b}{ax}$, $\frac{cd}{ax}$, and $\frac{x-a}{ax}$ have a common denominator.

127. Lowest Common Denominator. If the common denominator of several fractions is of the lowest degree possible, that is, if it contains no unnecessary factors, the fractions are said to have the *lowest common denominator* (L.C.D.).

Reduce $\frac{a}{bc}$ and $\frac{b}{cd}$ to equivalent fractions having the L.C.D.

The L.C.D. must contain all the factors of bc and cd, and no other factors; that is, it must be of the lowest possible degree.

The L.C.D. must therefore contain b, c, and d, and no other factors, and hence is bcd.

Evidently if we multiply both terms of $\frac{a}{bc}$ by d, and both terms of $\frac{b}{cd}$ by b, each fraction has the L.C.D. bcd.

Therefore the result is $\frac{ad}{bcd}$ and $\frac{b^2}{bcd}$.

The student should express fractions in lowest terms before beginning to find their L.C.D.

128. Finding the Algebraic Sum of Fractions. As in arithmetic, to add fractions we proceed as follows:

If the fractions have the same denominators, write the algebraic sum of the numerators over the common denominator.

If the fractions have different denominators, express the fractions as equivalent fractions having the L.C.D., and write the algebraic sum of the new numerators over the common denominator.

In either case reduce the resulting fraction to lowest terms.

Exercise 101. Addition and Subtraction of Fractions

Examples 1 to 7, oral

1. Add $\frac{a}{b}$ and $\frac{c}{b}$; $\frac{a}{x^2}$ and $\frac{b}{x^2}$; $\frac{a}{m}$ and $-\frac{b}{m}$.

2. Add $\frac{a+b}{x}$ and $\frac{a-b}{x}$; $\frac{a-b}{m}$ and $\frac{a-b}{m}$.

3. Add $\frac{1}{2}$ and $\frac{3}{4}$; $\frac{1}{2}$ and $\frac{5}{8}$; $\frac{3}{4}$ and $\frac{7}{8}$.

Add the following:

4. $\frac{a^2-b^2}{x+y}, \frac{a^2+b^2}{x+y}$.

5. $\frac{a^2-25}{x-y}, \frac{a^2+25}{x-y}$.

6. $\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}, -\frac{b}{2}$.

7. $\frac{a-b}{4}, \frac{b-c}{4}, \frac{c}{4}$.

8. Add $\frac{a+b}{a-b}$ and $\frac{2ab}{a^2-b^2}$. Check the result.

Add the following and check the results:

9. $\frac{x-y}{2}+\frac{y-x}{12}$.

10. $\frac{x+y}{4}-\frac{x-y}{8}$.

11. $\frac{2mn}{5}-\frac{m^2-2mn+n^2}{15}$.

12. $\frac{m^2-n^2}{5}+\frac{m^2+2mn+n^2}{10}$.

13. Show that $\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}$, and from this formula write a rule for adding two such fractions. Apply the formula to the addition of $\frac{1}{3}$ and $\frac{1}{4}$; of $\frac{1}{3}$ and $\frac{1}{5}$; of $\frac{1}{3}$ and $\frac{1}{8}$.

14. Show that $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$, and from this formula write a rule for subtraction in the case of two such fractions. Apply the formula to $\frac{1}{3}-\frac{1}{4}$; $\frac{1}{3}-\frac{1}{8}$; $\frac{1}{2}-\frac{1}{3}$.

15. Find the sum of $\frac{a}{b}+\frac{x}{y}$ and also of $\frac{a}{b}-\frac{x}{y}$. From the formulas write rules as in Exs. 13 and 14, illustrating each.

The student should see how valuable algebra is in enabling him to discover rules of this kind which can be used in commercial work.

Add the following and check the results:

16. $\frac{a}{3}+\frac{3a}{4}-\frac{5a}{12}$.

17. $\frac{x}{5}-\frac{2x}{3}+\frac{7x}{15}$.

18. $\frac{2a}{7}-\frac{a}{3}+\frac{5a}{21}$.

19. $\frac{4m}{5}-\frac{7m}{10}+\frac{4m}{15}$.

20. $\frac{3}{a}+\frac{4}{a^2}-\frac{5}{a^3}$.

21. $\frac{3}{a^3}-\frac{2}{3a^2}+\frac{1}{a}$.

22. $\frac{2}{3y}+\frac{3}{4y^2}-\frac{5}{12}$.

23. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

24. $\frac{a+b}{a-b}-\frac{a+b}{a^2-b^2}$.

25. $\frac{x}{x-y}-\frac{y}{x^2-y^2}$.

26. $\frac{a^2}{(a-1)^2}-\frac{a}{a-1}$.

27. $\frac{3}{a-3}+\frac{4}{a+3}$.

28. $\frac{5}{a-2}-\frac{3}{a+2}$.

29. $\frac{x}{x^2-y^2}+\frac{y}{x-y}$.

30. $\frac{x}{x^2-y^2}-\frac{y}{x+y}$.

31. $\frac{x+5}{x-5}-\frac{x-5}{x+5}$.

129. Mixed Expression. An algebraic expression which is the indicated algebraic sum of an integer and a fraction is called a *mixed expression.*

For example, $2\,a + b/c$ is a mixed expression. In arithmetic the plus sign is omitted between the integer and fraction in a mixed number, $2\frac{1}{3}$ meaning $2 + \frac{1}{3}$. In algebra $a\frac{b}{c}$ means $a \times \frac{b}{c}$, and hence the plus sign or the minus sign must be used in a mixed expression.

130. Reduction of a Fraction to a Mixed Expression. If the degree of the numerator of a fraction equals or exceeds that of the denominator, in a given letter, the fraction may be reduced to an integer or a mixed expression by dividing the numerator by the denominator.

Just as we reduce $\frac{5}{3}$ to $1\frac{2}{3}$ by dividing, so we may reduce $\frac{a^2 + b^2}{a + b}$ to $a - b + \frac{2\,b^2}{a + b}$. We cannot, however, reduce $\frac{x^2 + y^2}{a + b}$ to a mixed expression, because different letters are involved.

1. Reduce $\dfrac{4\,x^2 - 6\,xy + 3\,y^2}{2\,x - y}$ to a mixed expression.

Dividing the numerator by the denominator, we have

$$\frac{4\,x^2 - 6\,xy + 3\,y^2}{2\,x - y} = 2\,x - 2\,y + \frac{y^2}{2\,x - y}.$$

2. Reduce $\dfrac{a^3 + 3\,a^2b - 6\,ab^2 + 2\,b^3}{a^2 + 2\,ab + b^2}$ to a mixed expression.

Dividing, $\dfrac{a^3 + 3\,a^2b - 6\,ab^2 + 2\,b^3}{a^2 + 2\,ab + b^2} = a + b$, with remainder $-9\,ab^2 + b^3$.

We may therefore write the result in the form

$$a + b + \frac{-9\,ab^2 + b^3}{a^2 + 2\,ab + b^2}, \text{ or } a + b - \frac{9\,ab^2 - b^3}{a^2 + 2\,ab + b^2}, \text{ or } a + b - \frac{b^2(9\,a - b)}{(a + b)^2},$$

the last form being the most convenient form for computation.

The work may be checked by substituting any valne for the letters that will not make the denominator zero. For example, if $a = 1$ and $b = 1$, we obtain 0 for the original fraction and for the result.

Exercise 102. Reduction to Mixed Expressions

Examples 1 to 11, oral

1. Reduce to integers: $\frac{10}{2}$; $\frac{10\,a^2}{2}$; $\frac{10\,a^2}{2\,a}$; $\frac{a^2b}{ab}$; $\frac{(a+b)^2}{a+b}$.

2. Reduce to integers: $\frac{a^2-b^2}{a-b}$; $\frac{a^2-b^2}{a+b}$; $\frac{a^2+2\,ab+b^2}{a+b}$.

3. Reduce to mixed expressions: $\frac{x+y}{x}$; $\frac{x-y}{x}$; $\frac{ab+c}{a}$.

Reduce to integral or mixed expressions:

4. $\frac{ax-ay}{a}$.
5. $\frac{x^4-y^4}{x^2+y^2}$.
6. $\frac{abx+aby}{ab}$.
7. $\frac{abx+1}{ab}$.
8. $\frac{mn^2+3}{mn}$.
9. $\frac{abc+c}{abc}$.
10. $\frac{m^6-n^6}{m^3-n^3}$.
11. $\frac{a+b}{a-b}$.

12. Reduce $\frac{a^2+3\,ab+4\,b^2}{a+b}$ to a mixed expression, and check the result by letting $a=2$ and $b=3$.

Reduce the following to integral or mixed expressions:

13. $\frac{x^2+3\,xy+y^2}{x+y}$.
14. $\frac{x^2-3\,xy-y^2}{x-y}$.
15. $\frac{x^2+15\,x+56}{x+7}$.
16. $\frac{x^2-15\,x+56}{x-8}$.
17. $\frac{a^2b^2+3\,ab+2}{ab+2}$.
18. $\frac{p^2+4\,pq+q^2}{p+q}$.
19. $\frac{m^3+3\,m^2n+3\,mn^2+n^3}{m^2+n^2+2\,mn}$.
20. $\frac{m^3+3\,m^2n+4\,mn^2+5\,n^3}{m\,(m+2\,n)+n^2}$.
21. $\frac{a^3+a^2b+ab^2+b^3}{a+b}$.
22. $\frac{x^3+3\,x^2y-xy^2-y^3}{x+y}$.
23. $\frac{a^3+b^3+c^3-3\,abc}{a+b+c}$.
24. $\frac{x^4+x^2y^2+y^4}{x^2+xy+y^2}$.

131. Reduction of a Mixed Expression to a Fraction. Since the value of an expression is not changed if it is both multiplied and divided by the same expression, it follows that $a = \frac{ab}{b}$, and $a + \frac{a}{b} = \frac{ab + a}{b}$. Therefore

To reduce a mixed expression to a fraction, multiply the integral part by the denominator, to the product add the numerator, and under the result write the denominator.

This is practically the same as adding two fractions. For example, $a + \frac{c}{d} = \frac{ad}{d} + \frac{c}{d} = \frac{ad + c}{d}$.

Reduce $a + 3b - \frac{a^2 - 2ab + 4b^2}{a - 3b}$ to a fraction.

Reduce $a + 3b$ to a fraction with $a - 3b$ for its denominator. Then

$$\frac{a^2 - 9b^2}{a - 3b} - \frac{a^2 - 2ab + 4b^2}{a - 3b} = \frac{a^2 - 9b^2 - a^2 + 2ab - 4b^2}{a - 3b} = \frac{2ab - 13b^2}{a - 3b}.$$

Exercise 103. Reduction of Mixed Expressions to Fractions

Examples 1 to 3, oral

1. Reduce to improper fractions: $2\frac{3}{8}$; $4\frac{5}{8}$; $9\frac{7}{8}$.

2. Reduce to fractions: $a + \frac{b}{c}$; $a - \frac{b}{c}$; $a + b + \frac{c}{d}$.

3. Reduce to fractions: $a + \frac{ac}{b + c}$; $a - \frac{ac}{b + c}$; $a + b + \frac{c}{a - b}$.

Reduce each of the following to a fraction:

4. $a + b + \frac{a^2 + b^2}{a - b}$.

5. $x + 2y + \frac{4y^2}{x - 2y}$.

6. $a^2 + ab + b^2 + \frac{2b^3}{a - b}$.

7. $\frac{a^2 + b^2}{a - b} + b + 4a$.

8. $a^2 - ab + b^2 - \frac{b^3}{a + b}$.

9. $\frac{x^2 + 2xy - y^2}{x + y} + x + y$.

132. Multiplication of Fractions. We multiply fractions in algebra in the same manner as in arithmetic, and hence the process need not be explained at length.

To multiply a fraction by an integral expression, multiply the numerator by the integral expression, writing the product over the denominator.

To multiply a fraction by a fraction, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Cancel factors common to any numerator and any denominator.

Exercise 104. Multiplication of Fractions

Examples 1 to 9, oral

Multiply the following:

1. $b \times \frac{a}{b}$.

2. $4 \times \frac{a}{4\,b}$.

3. $4\,b \times \frac{a}{4\,b}$.

4. $mn \cdot \frac{a}{mn}$.

5. $p^2q \cdot \frac{1}{pq}$.

6. $x^2y^2 \cdot \frac{1}{xy}$.

7. $\frac{3^2}{5^3} \cdot \frac{5^2}{3^2}$.

8. $\frac{m}{n} \cdot \frac{n}{m}$.

9. $\frac{m}{n^2} \cdot \frac{n}{m^2}$.

10. Find the product of $\frac{2\,ax^2}{3\,by^2} \cdot \frac{3\,cz^2}{4\,ax^2} \cdot \frac{4\,by^2}{5\,cz^2}$, and check.

Multiply the following and check:

11. $\frac{ab^2}{c^2d} \cdot \frac{bc^2}{d^2a} \cdot \frac{cd^2}{a^2b}$.

12. $\frac{a^2-b^2}{a^2+b^2} \cdot \frac{4\,a}{a+b}$.

13. $\frac{a^2-b^2}{a^2+b^2} \cdot \frac{2\,a}{3\,b+3\,a}$.

14. $\frac{(2\,a)^2}{b} \cdot \frac{(3\,b)^2}{c} \cdot \frac{(4\,c)^2}{a}$.

15. $\frac{4\,x^3+8\,x^2}{x-3} \cdot \frac{9\,x-x^3}{2+x}$.

16. $\frac{(a+b)^2}{a-b} \cdot \frac{a^2-b^2}{a^2+b^2}$.

133. Reciprocal. If the product of two expressions is 1, each expression is called the *reciprocal* of the other.

Since $\frac{a}{b} \cdot \frac{b}{a} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and conversely.

134. Division of Fractions. Just as in arithmetic,

To divide by a fraction, multiply by its reciprocal.

Exercise 105. Division of Fractions

Examples 1 to 8, oral

1. Divide $\frac{a}{2}$ by $\frac{a}{4}$; $\frac{a}{4}$ by $\frac{a}{2}$; $\frac{2}{a}$ by $\frac{4}{a}$; $\frac{4}{a}$ by $\frac{2}{a}$; $\frac{4}{a}$ by $\frac{a}{2}$.

2. Divide $\frac{a+b}{x}$ by $\frac{a-b}{x}$; $\frac{a+b}{2x}$ by $\frac{a-b}{2x}$; $\frac{a+b}{x+y}$ by $\frac{a-b}{x+y}$.

Divide the following as indicated:

3. $\frac{ab}{xy} \div \frac{cd}{xy}$.

4. $\frac{abc}{xy} \div \frac{bcd}{xy}$.

5. $\frac{2x}{y} \div \frac{3x}{y}$.

6. $\frac{2x}{y} \div \frac{3xy}{y^2}$.

7. $\frac{m}{n} \div \frac{n}{m}$.

8. $\frac{ab}{cd} \div \frac{cd}{ab}$.

9. Divide $\frac{15\,a^2b}{16\,c^2d}$ by $\frac{3\,ab^2}{4\,cd^2}$, and check the result by letting $a = 2$, $b = 3$, $c = 4$, $d = 5$.

Divide the following as indicated:

10. $\frac{25\,abc}{32\,xyz} \div \frac{15\,a^2b^2c^2}{28\,x^2y^2z^2}$.

11. $\frac{17\,x^2yz}{24\,m^3n^2} \div \frac{34\,xy^2z^3}{75\,m^4n^5}$.

12. $\frac{35\,a^4b^5c^6}{27\,x^4y^5z^6} \div \frac{28\,a^3b^2c}{81\,x^2y^3z^4}$.

13. $\frac{25\,m^3n^4z}{33\,a^4b^5c^6} \div \frac{35\,mnz^4}{44\,a^3b^3c^3}$.

14. $\frac{a^2+b^2}{a-b} \div \frac{a^2+b^2}{a+b}$.

15. $\frac{a^2-b^2}{a^2+b^2} \div \frac{a+b}{a^2+b^2}$.

16. $\frac{a^4-b^4}{a^4+b^4} \div \frac{a^2-b^2}{a^2+b^2}$.

17. $\frac{m^2+n^2}{m^2-n^2} \div \frac{2\,m^2+2\,n^2}{m-n}$.

Exercise 106. Formulas

Examples 1 to 28, oral

Using $\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}$ *and* $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$ *as formulas, perform the following additions and subtractions:*

1. $\frac{1}{2}+\frac{1}{3}$.	**5.** $\frac{1}{2}-\frac{1}{3}$.	**9.** $\frac{1}{4}+\frac{1}{8}$.	**13.** $\frac{1}{4}-\frac{1}{8}$.
2. $\frac{1}{2}+\frac{1}{4}$.	**6.** $\frac{1}{2}-\frac{1}{4}$.	**10.** $\frac{1}{3}+\frac{1}{8}$.	**14.** $\frac{1}{3}-\frac{1}{8}$.
3. $\frac{1}{2}+\frac{1}{8}$.	**7.** $\frac{1}{2}-\frac{1}{8}$.	**11.** $\frac{1}{4}+\frac{1}{5}$.	**15.** $\frac{1}{4}-\frac{1}{5}$.
4. $\frac{1}{3}+\frac{1}{4}$.	**8.** $\frac{1}{3}-\frac{1}{4}$.	**12.** $\frac{1}{2}+\frac{1}{5}$.	**16.** $\frac{1}{2}-\frac{1}{5}$.

Using $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ *and* $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$ *as formulas, perform the following additions and subtractions:*

17. $\frac{2}{3}+\frac{3}{4}$.	**20.** $\frac{3}{4}-\frac{2}{3}$.	**23.** $\frac{5}{6}+\frac{2}{3}$.	**26.** $\frac{5}{6}-\frac{2}{3}$.
18. $\frac{1}{2}+\frac{3}{5}$.	**21.** $\frac{3}{5}-\frac{1}{2}$.	**24.** $\frac{3}{5}+\frac{1}{4}$.	**27.** $\frac{3}{5}-\frac{1}{4}$.
19. $\frac{3}{4}+\frac{2}{5}$.	**22.** $\frac{3}{4}-\frac{2}{5}$.	**25.** $\frac{2}{5}+\frac{2}{3}$.	**28.** $\frac{2}{3}-\frac{1}{7}$.

Examining the problem $a \div \frac{100}{p}=\frac{ap}{100}$, we see that any number a is divided by $\frac{1}{p}$ of 100 by multiplying a by p and moving the decimal point two places to the left. Thus $84 \div 12\frac{1}{2}=\frac{84 \times 8}{100}$, or 6.72. In like manner perform the indicated operations in Exs. 29–40.

Divide the following as indicated:

29. $65 \div 12\frac{1}{2}$.	**33.** $76 \div 25$.	**37.** $79 \div 33\frac{1}{3}$.
30. $4.83 \div 16\frac{2}{3}$.	**34.** $85 \div 33\frac{1}{3}$.	**38.** $436 \div 12\frac{1}{2}$.
31. $45 \div 25$.	**35.** $94 \div 8\frac{1}{3}$.	**39.** $57 \div 16\frac{2}{3}$.
32. $66 \div 50$.	**36.** $150 \div 12\frac{1}{2}$.	**40.** $85 \div 8\frac{1}{3}$.

41. Simplify $a \div \frac{1000}{p}$, and write the rule for dividing any number a by $\frac{1}{p}$ of 1000.

42. If the correct value of a fraction is $\frac{a}{b}$ and an estimate of its value is given as $\frac{a-x}{b+y}$, by how much is the estimate too small? Use particular values for the letters involved, such as $a=3$, $b=5$, $x=1$, and $y=1$, and substitute in your result as a check.

43. If the correct value of a fraction is $\frac{a}{b}$ and an estimate is given as $\frac{a+x}{b-y}$, by how much is the estimate too large? Illustrate by using particular numbers for a, b, x, and y.

44. If x and y represent the greatest possible error in estimating dividend a and divisor d respectively, the greatest possible error of the result of division is approximately $\frac{1}{2}$ of $\frac{a+x}{d-y}-\frac{a-x}{d+y}$. Find its value.

45. Using the result of Ex. 44, find the greatest possible error of the answer obtained by correctly dividing 4300 by 510, if the greatest possible error of the dividend is 50 and that of the divisor is 5.

46. The formula for the present value of an annuity is $A=\frac{R}{r}\left[1-\frac{1}{(1+r)^m}\right]$. Express this formula so that the part in brackets shall be an improper fraction.

47. If a series of items whose average is to be found is represented by $m_1 \pm e_1$, $m_2 \pm e_2$, $m_3 \pm e_3$, $m_4 \pm e_4$, and $m_5 \pm e_5$, the greatest possible average will be obtained when all the errors (e's) are positive, and the least possible average when all the errors are negative. Find the fraction representing the average in each case, and separate each result into two fractions, one in which the numerator consists of m's and the other in which the numerator consists of e's.

Exercise 107. Review of Chapter IX

1. Reduce to lowest terms: $\frac{ax}{ay}$, $\frac{2\times 3}{2\times 5}$, $\frac{2^3\times 3^5}{2^3\times 7^2}$, $\frac{3\times 1.05^3}{2\times 1.05^4}$.

2. The ratio of the amounts of two principals, p_1 and p_2, in t years at r per cent is found by dividing the equations $a_1 = p_1(1+rt)$ and $a_2 = p_2(1+rt)$, member by member. Find the value of $\frac{a_1}{a_2}$ in terms of p_1 and p_2.

3. If contractor A can do $\frac{1}{m}$ of a piece of work in 1 da., and contractor B can do $\frac{1}{k}$ of the work in 1 da., what part of the work can they do in 1 da. working together?

4. What part of the work in Ex. 3 would be done in 1 da. if A could do all the work in 3 da. and B in 5 da.?

Substitute 3 for m and 5 for k in the result of Ex. 3.

5. Using $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ twice, add $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{3}{8}$.

6. Divide \$84.65 by \$12.50, making use of the formula $a \div \frac{100}{p} = \frac{ap}{100}$.

7. Reduce $\frac{3^2\times 5\times 7}{3\times 7}$ to an integer.

8. Simplify $\frac{1}{b}\times\left(\frac{c}{d}+\frac{e}{f}\right)$ and $\frac{1}{2}\times\left(\frac{2}{3}+\frac{3}{4}\right)$.

9. Simplify $\frac{a^2-b^2}{(a+b)^2}\div\frac{a-b}{a+b}$ and $\frac{3}{4}\div\frac{5}{2}$.

10. If i_1 is the interest on p for 1 da. at $4\frac{1}{2}\%$, and i_2 is the interest on p for 1 da. at 3%, find the value of $\frac{i_1}{i_2}$.

11. If $i=\frac{prt}{360}$ and $i'=\frac{prt}{365}$, find the values of $\frac{i}{i'}$ and $\frac{i'}{i}$.

Exercise 108. Review of Chapters I–IX

1. The formula for the area of a triangle is $a = \frac{bh}{2}$. If $b = 2$, what is the value of a?

2. The area of a circle is expressed by the formula $a = \pi r^2$. If $r = 1/\pi$, what is the value of a?

3. Find the sum of $34 \times \$1.25$, $45 \times \$1.25$, $15 \times \$1.25$, and $66 \times \$1.25$. What algebraic principle did you use?

4. If 184 is $\frac{x}{100}$ of 450, find the value of x.

5. What per cent of 1440 is 92?

6. If 65 doz. eggs are packed in containers costing \$3.90 and sold for \$26.65, how much is received per dozen?

7. By how much does $2x - 1$ exceed $3 - 2x$?

8. Using the principle $ab = ba$, find 72% of \$125.

9. Using the principle $\frac{abc}{ab} = \left(\frac{abc}{a}\right) \div b$, divide \$162.45 by 45.

10. Three companies, A, B, and C, enter into a pooling agreement with a common sales agency, with the understanding that the agency shall assign the business to the companies in proportion to their average total business for the last 5 yr. These averages were \$2,800,000, \$3,500,000, and \$4,000,000 respectively. What portion of an \$800,000 order should be assigned to each company?

11. If goods cost \$60, how should they be marked so that a discount of 20% of the marked price can be allowed and a profit of 15% of the cost be realized?

12. From the equation $i = prt$ derive a rule based on 360 da. to 1 yr. for finding interest for 1 da. at $2\frac{1}{4}\%$. Use this rule to find the interest on \$8400 for 32 da. at $2\frac{1}{4}\%$.

CHAPTER X

FRACTIONAL EQUATIONS

135. Fractional Equations. An equation containing a fraction or several fractions is called a *fractional equation.*

Multiplying both members of a fractional equation by such an expression as shall leave no fractions in the equation is called *clearing the equation of fractions.*

If $\frac{x}{a}+\frac{x}{b}=ab,$

multiplying by ab, $bx+ax=a^2b^2$, an equation cleared of fractions.

To clear an equation of fractions, multiply both members of the equation by the lowest common denominator of the fractions.

If a fraction is preceded by a minus sign, change the sign of every term in the numerator when the denominator is removed.

1. Solve the equation $\frac{x}{2}+\frac{3}{a}-4=\frac{1}{a}-\frac{1}{4}.$

Multiplying by $4a$, the L. C. D. of the fractions,

$$2ax+12-16a=4-a.$$

Subtracting 12, $2ax-16a=-8-a.$

Subtracting $-16a$, $2ax=15a-8.$

Dividing by $2a$, $x=\frac{15a-8}{2a}.$

2. Solve the equation $\frac{x+1}{x-1}=\frac{x-1}{x-2}.$

Multiplying by the L. C. D., $x^2-x-2=x^2-2x+1.$

Solving, $x=3.$

Exercise 109. Fractional Equations

Examples 1 to 4, oral

1. Clear of fractions: $\frac{x}{a}=b$; $\frac{x}{2a}=\frac{a}{2}$; $\frac{x}{3}+\frac{1}{4}=1$.

2. Solve $\frac{x}{3}+1=2$; $\frac{x}{4}-1=2$; $\frac{x}{5}-2=2$.

3. Solve $\frac{2x}{3}=4$; $\frac{3x}{4}=6$; $\frac{4x}{5}=8$; $\frac{5x}{7}=10$.

4. Solve $\frac{x+1}{2}=3$; $\frac{x-1}{2}=5$; $\frac{x+1}{3}=6$; $\frac{x-1}{5}=2$.

5. Given $i=prt$, solve for p in terms of i, r, t, and write the rule in interest derived from the resulting formula.

6. As in Ex. 5, solve the equation $i=prt$ for r and for t, writing the rule in each case.

7. What sum will yield \$3600 interest in 4 yr. at 4%?

8. In what time will \$8000 earn \$2000 interest at 5%?

9. At what rate will \$5000 yield \$600 interest in 2 yr.?

10. Given the formula $a=p(1+rt)$, solve for p, then for r, and then for t. Write the rule derived from each formula.

Solve the following equations:

11. $\frac{x+1}{x-1}=\frac{5}{3}$.

12. $\frac{x-1}{x+1}=\frac{3}{4}$.

13. $\frac{x+2}{x-2}=1\frac{2}{9}$.

14. $\frac{x-2}{x+2}=\frac{3}{7}$.

15. $\frac{x-3}{x+3}=0.4$.

16. $\frac{x+7}{x-3}=3$.

17. $\frac{x+\frac{1}{2}}{x+\frac{3}{4}}=\frac{3}{4}$.

18. $\frac{x-5}{x-8}=4$.

19. $\frac{x+3}{x-5}=9$.

20. $\frac{x-3}{x-7}=3$.

21. $\frac{x+1}{x-1}=\frac{2x-5}{2x-7}$.

22. $\frac{x+2}{x-3}=\frac{2x-5}{2x-10}$.

136. Combining Terms before Clearing. It is often better to combine terms before clearing an equation of fractions.

1. Solve the equation $\frac{x+1}{2a}+\frac{2x}{3a}=\frac{1}{2a}+\frac{x}{3a}+5.$

Here it is apparently advisable to unite the fractions with the same denominators before clearing of fractions.

Subtracting $\frac{1}{2a}$ and $\frac{x}{3a}$,

$$\frac{x+1-1}{2a}+\frac{2x-x}{3a}=5,$$

or

$$\frac{x}{2a}+\frac{x}{3a}=5.$$

Combining,

$$\frac{5x}{6a}=5.$$

Dividing by 5,

$$\frac{x}{6a}=1.$$

Multiplying by $6a$,

$$x=6a.$$

2. Solve the equation $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

Combining the fractions in the respective members,

$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-5)(x-7)-(x-6)^2}{(x-6)(x-7)}.$$

Simplifying the numerators,

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}.$$

Multiplying by -1, and then clearing of fractions,

$$(x-6)(x-7)=(x-2)(x-3).$$

Expanding, $x^2-13x+42=x^2-5x+6.$

Therefore $-8x=-36,$

and $x=4\frac{1}{2}.$

We might first reduce to mixed fractions, thus:

$$1+\frac{1}{x-2}-1-\frac{1}{x-3}=1+\frac{1}{x-6}-1-\frac{1}{x-7}.$$

Exercise 110. Fractional Equations

Examples 1 to 3, oral

1. Solve the equations $x+\frac{3}{2}-\frac{1}{2}=2$; $x+\frac{4}{5}=3-\frac{1}{5}$.
2. Solve the equations $x+\frac{2}{3}-\frac{1}{3}=2\frac{1}{3}$; $x+\frac{3}{4}+\frac{1}{4}=4$.
3. Solve the equations $x+\frac{7}{8}-\frac{3}{8}=5\frac{1}{2}$; $x+\frac{5}{8}+\frac{3}{8}=17$.

Solve the following equations:

4. $\dfrac{3x-5}{5x-5}+\dfrac{5x-1}{7x-7}+\dfrac{x-4}{x-1}=2.$

First factor the first two denominators.

5. $\dfrac{8x+2}{x-2}-\dfrac{2x-1}{3x-6}+\dfrac{3x+2}{5x-10}+5=15.$

6. $\dfrac{3x-1}{2x-6}+\dfrac{5x-7}{3x-9}+\dfrac{7x+1}{4x-12}+14=25.$

7. $\dfrac{3x}{2x-2}+\dfrac{5}{6(x-1)}-\dfrac{15x-7}{9(x+1)}=\dfrac{17x-x^2}{6(x^2-1)}+\dfrac{71}{18(x^2-1)}.$

8. $\dfrac{4}{x+1}+\dfrac{x}{1-x}-\dfrac{1}{x-1}=\dfrac{x^2-3}{1-x^2}.$

Write the first fraction $\dfrac{4}{1+x}$ and the third fraction $+\dfrac{1}{1-x}$.

Solve the following equations and check the results:

9. $\dfrac{4}{x+2}+\dfrac{7}{3+x}-\dfrac{37}{x(x+5)+6}=0.$

10. $\dfrac{1}{x-1}-\dfrac{1}{x-2}=\dfrac{1}{x-3}-\dfrac{1}{x-4}.$

11. One fifteenth of $6x+7$, divided by x, is equal to the quotient of $2x-2$ divided by $5x-6$. Find the value of x.

12. One fifteenth of $6x+1$, diminished by the quotient of $2x-4$ divided by $7x-16$, is equal to one fifth of $2x-1$. Find the value of x.

137. Equations used in Formulas. 1. From the formula for the amount $a = p + prt$, derive a formula for the time, t.

From the equation $a = p + prt$

we may obtain the value of t as follows:

$$\frac{a - p}{pr} = t.$$

That is, if we divide the difference between the amount and the principal by the product of principal and rate, we obtain the number of years required for the given principal to produce the given interest at the given rate.

2. From $c = 2\pi r$ derive a formula for the radius.

Dividing each member of the given equation by 2π, we have

$$\frac{c}{2\pi} = r.$$

That is, if we divide the circumference by $2 \times 3\frac{1}{7}$ we shall obtain the radius of the circle.

3. Given the formula $v = \pi r^2 h$, derive a formula for r.

Dividing by πh, $\quad \frac{v}{\pi h} = r^2.$

Taking the square root, $\quad \sqrt{\frac{v}{\pi h}} = r.$

In such a case we can only indicate the square root. In practice the square root of a number is usually found from tables.

138. Use of Primes and Subscripts. If we wish to use a formula in which occur radii of two different circles, we may represent one radius by r and the other by r'.

For example, $c = 2\pi r$, and $c' = 2\pi r'$; then $c - c' = 2\pi(r - r')$.

It is often convenient to use r' and r'' (read "r second") or, preferably, r_1 and r_2 (read "r sub-one" and "r sub-two"). We read r_1^2 "r sub-one square" or "the square of r sub-one."

Exercise 111. Formulas as Equations

Examples 1 to 6, oral

1. Solve $a = \pi r^2$ for r.

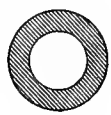
2. Solve $a = \frac{1}{2} bh$ for h; also for b.

3. Solve $v = bwl$ for b; also for w; also for l.

4. Solve $a = \frac{1}{2} h(b + b')$ for h. How do you solve for b?

5. Solve $v = \pi r^2 h$ for h; also for r^2.

6. Solve $a = p(1 + rt)$ for p. How do you solve for r? also for t? also for p?

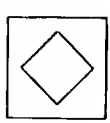


7. The area of a ring is indicated by the formula $a = \pi (r_1 + r_2)(r_1 - r_2)$. Find a formula for r_1; for r_2.

8. In Ex. 7 suppose that $a = 286$, $r_2 = 10$, and $\pi = 3\frac{1}{7}$, find the value of r_2.

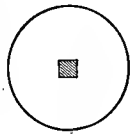
9. In Ex. 7 suppose that $a = 506$, $r_2 = 12$, and $\pi = 3\frac{1}{7}$, find the value of r_1.

10. If from a square of side s_1 there is cut out a smaller square of side s_2, the area of the part that remains is given by the formula $a = (s_1 + s_2)(s_1 - s_2)$. Find a when $s_1 = 11$, $s_2 = 9$; also find s_2 when a and s_1 are given.



11. In connecting two wheels by a belt it is often required to know the difference in the two circumferences. This is given by the formula $D = 2\pi (r_1 - r_2)$. Suppose that D and r_2 are known, find r_1.

12. If from a circle of radius r there is cut a square of side s, the area that remains is given by the formula $a = \pi r^2 - s^2$. If $r = 7$ and $s = 4$, find a. If a and s are known, find r. If a and r are known, find s.



The following formulas will be met by the student in his subsequent work in mathematics or physics. Solve as directed:

13. $s = vt$; find t.

14. $s = vt$; find v.

15. $d = rt$; find r.

16. $d = rt$; find t.

17. $s = \frac{1}{2}gt^2$; find g.

18. $s = \frac{1}{2}gt^2$; find t.

19. $W_1L_2 = W_2L_1$; find L_1.

20. $W_1L_2 = W_2L_1$; find W_2.

21. $W = Fs$; find s.

22. $V_1P_1 = V_2P_2$; find P_2.

23. $V_1P_1 = V_2P_2$; find P_1.

24. $v_2t = v_1t + n$; find v_1.

25. $v_2t = v_1t + n$; find t.

26. $v_2t = v_1t + n$; find n.

27. $l = Kl'(t' - t)$; find K.

28. $D(w_1 - w_2) = w_1$; find w_2.

29. Given $s = \frac{1}{2}n(a + l)$. Solve for n; for a; for l.

30. Given $s = \dfrac{rl - a}{r - 1}$. Solve for r; for l; for a.

31. Given $a = \frac{1}{2}h(b + b')$. Solve for b; for b'.

32. Given $C = \dfrac{en}{R + nr}$. Solve for e; for n; for R; for r.

33. In physics is found the formula $v = at + \frac{1}{2}gt^2$ which relates to moving bodies. Solve for a; for g.

34. The volume of a sphere is expressed by the formula $v = \frac{4}{3}\pi r^3$. Find a formula for r^3.

35. If the side of the square base of a pyramid is s and the height of the pyramid is h, the volume is $v = \frac{1}{3}hs^2$. From this formula find a formula for h; for s^2; for s.

36. The centigrade thermometer is commonly used in scientific work. The number of degrees in the centigrade scale can be found in our common (Fahrenheit) scale by the formula $C = \frac{5}{9}(F - 32)$, where C is the number of degrees centigrade and F the number of degrees Fahrenheit. Given $F = 70$, find the value of C. Also find F in terms of C.

Given the following formulas and using $3\frac{1}{7}$ for π wherever it occurs, find the value of the missing letter in each case:

PARALLELOGRAM $a = bh$

	a	b	h
37.		$7\frac{1}{2}$	$9\frac{1}{2}$
38.		$4\frac{3}{4}$	$2\frac{1}{2}$
39.	75	15	
40.	161	23	
41.	319	29	
42.	341		11
43.	403		13
44.	437		19

CIRCLE $c = 2\pi r$

	c	r
45.		7
46.		14
47.		$3\frac{1}{2}$
48.		21
49.	44	
50.	88	
51.	132	
52.	77	

SPHERE $s = 4\pi r^2$

	s	r
53.		3
54.		5
55.		6
56.		10
57.	154	
58.	616	
59.	1386	
60.	2464	

CYLINDER $v = \pi r^2 h$

	v	r	h
61.		2	7
62.		7	2
63.		2	14
64.		7	15
65.	4400	10	
66.	1232	14	

CIRCLE $a = \pi r^2$

	a	r
67.		2
68.		4
69.		7
70.	154	
71.	616	
72.	1386	

SPHERE $v = \frac{4}{3}\pi r^3$

	v	r
73.		1
74.		2
75.		3
76.	$113\frac{1}{7}$	
77.	$1437\frac{1}{3}$	
78.	$4\frac{4}{21}$	

Given the area of a triangle $a = \frac{1}{2}bh$, find b when:

79. $a = 15$, $h = 5$.

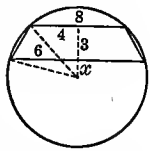
80. $a = 17.5$, $h = 5$.

Given the area of a triangle $a = \frac{1}{2}bh$, find h when:

81. $a = 8.25$, $b = 5.5$.

82. $a = 10\frac{6}{16}$, $b = 3\frac{1}{8}$.

83. The bases of a trapezoid inscribed in a circle are 12 in. and 8 in. respectively, and the altitude is 3 in. Find the distance from the center to the lower base, and find the radius of the circle.



Show that $x^2 + 6^2 = (x + 3)^2 + 4^2$.

84. A caterer prepares a dinner for 50 persons at a cost of \$50 to himself. How much must he charge per plate to be sure of making a profit of \$17.80 in case only 45 persons come to the dinner, counting the unused food as worth $20\,¢$ per plate?

85. A boy walked from A to B at the rate of $3\frac{1}{2}$ mi. an hour. He returned immediately on horseback at the rate of $4\frac{2}{3}$ mi. an hour. Upon his return he found that he had been gone $3\frac{1}{2}$ hr. How far is it from A to B?

86. The Atlantic and Southern Machine Co. fixes the price of its shell crushers so as to realize 40% profit on the cost of production. The company submits a bid to furnish 300 machines for \$3827, f.o.b. the point of delivery. The freight would be \$47. Find the cost of producing one machine.

The symbol "f.o.b." means "free on board," that is, charges prepaid until the goods are on the freight train or boat at the place specified.

87. The F. H. Hobbs Company buys boots from the manufacturers at \$30 per dozen pairs. If the discounts from the list price amount to 23% on the average, and the firm wishes to realize a profit of 20% on the cost, what list price must it quote on the boots?

88. A retailer has found that the average cost of doing business is 20% of his gross sales for the year. His profits have averaged 25% of the gross sales. At what price must he sell stoves that cost him \$18 each to cover these items at the rates mentioned?

139. Miscellaneous Problems. Consider the following illustration: The sum of two fractions is $\frac{3}{4}$, and one of them is known to be $\frac{5}{7}$ of the other. Find the fractions.

Let	$x =$ the larger fraction.
Then	$\frac{5}{7}x =$ the smaller fraction.
Therefore	$x + \frac{5}{7}x = \frac{3}{4}$.
Combining,	$\frac{12}{7}x = \frac{3}{4}$.
Dividing by $\frac{12}{7}$,	$x = \frac{7}{16}$.
Then	$\frac{5}{7}x = \frac{5}{7} \times \frac{7}{16} = \frac{5}{16}$.

Therefore the required fractions are $\frac{7}{16}$ and $\frac{5}{16}$.

Check. Substituting in the statement of the problem,

$$\frac{5}{16} + \frac{7}{16} = \frac{12}{16} = \frac{3}{4}, \text{ and } \frac{5}{7} \text{ of } \frac{7}{16} = \frac{5}{16}.$$

Exercise 112. Problems

1. If 1 more than a certain number is multiplied by the number, the product is 3 more than the square of the number. What is the number?

2. If 2 less than a certain number is multiplied by the number, the product is 4 less than the square of the number. What is the number?

3. If a certain number increased by 1 is multiplied by the number increased by 2, the product is the sum of 8 and the square of the number. What is the number?

4. If a certain number increased by 2 is multiplied by the number increased by 3, the product is 16 more than the square of the number. What is the number?

5. Find two consecutive numbers, x and $x+1$, such that $\frac{1}{2}$ of the smaller exceeds $\frac{1}{3}$ of the larger by 5.

6. Find three consecutive numbers such that their sum is 99; three others such that their sum is 159.

7. How much water must be added to a gallon of a 20% solution of alcohol to reduce it to a 10% solution?

8. How much water must be added to 32 gal. of alcohol, 95% pure, to reduce it to a 75% solution?

9. How much water must be added to 24 gal. of alcohol, 85% pure, to reduce it to a 60% solution?

10. How many pounds of water must be added to 160 lb. of a 5% solution of salt to reduce it to a 4% solution? How many gallons, allowing 8.35 lb. of water to a gallon?

11. How much water must be added to a pint of a 5% solution of a certain medicine to reduce it to a 1% solution?

12. How much water must be added to a barrel of vinegar, 85% pure, to reduce it to a 50% solution?

13. How much pure alcohol must be added to a gallon of alcohol, 20% pure, so that the mixture shall be 30% pure?

14. In 90 oz. of an alloy of silver and copper there are 6 oz. of silver. How much copper must be added so that 50 oz. of the new alloy may contain 2 oz. of silver?

15. In 3 oz. of a mixture of water and listerine there is 1 oz. of listerine. How much listerine must be added to make the new solution contain 75% listerine?

16. Our silver coins contain 90% silver. How much silver must be melted with 100 oz. of an alloy containing 80% silver to bring the alloy to the standard required for our coinage?

17. In Ex. 16 how much silver must be melted with 200 oz. of an alloy containing 75% silver to bring the alloy to the 90% standard required for our coinage?

18. Air consists by volume of four parts of nitrogen and one part of oxygen. How many cubic feet of oxygen are there in a schoolroom 30 ft. by 40 ft. by 10 ft.? How high must a room 30 ft. by 40 ft. be to contain 3600 cu. ft. of oxygen?

19. It is desired to lay out a circular parade ground having an area of 22 A. Express in rods as a square root the radius to be used in describing the circle.

20. Referring to Ex. 19, derive a formula for finding the number of rods to be used as radius for describing the circle having an area of a acres.

21. Change Ex. 20 so as to find the number of feet to be used as radius.

22. A merchant does a business of \$120,000 in one year and \$150,000 the next year. Find the rate of increase and write a formula which you can use for all similar cases.

23. A ditch for draining a large piece of land in three towns cost \$30,000. The towns agree to pay according to the length of the ditch within their respective limits. Town A has the greatest length, B has $\frac{2}{3}$ as much as A, and C has $\frac{3}{4}$ as much as A. How much should each town pay?

24. Two partners agree to divide their profits by allowing the first 30% more than the second because he owns the building. How should they divide a profit of \$12,000?

25. Two partners agree to divide their profits by allowing the first r per cent more than the second. How should they divide a profit of P dollars? Check the formula by applying it to Ex. 24.

26. Three partners, A, B, and C, own a business and divide the profits according to the number of shares of capital stock owned by each. A owns 50% more shares than B, and C owns 20% less than B. How shall they divide \$15,000?

27. The cost of building a road, R dollars, is to be borne jointly by the state, the county, and the town. The state agrees to pay S per cent of the cost, the county C per cent, and the town the remainder. What is the share of each?

140. Monomial and Polynomial Denominators. When an equation contains fractions some of which have monomial denominators, it is usually advisable to clear the equation of these fractions first.

Solve $x^2+\frac{1}{2}+\frac{9x-7}{6(3x+11)}=\frac{3x^2+2}{3}$.

Multiplying by 6, $6x^2+3+\frac{9x-7}{3x+11}=6x^2+4$.

Subtracting $6x^2$ and 3, $\frac{9x-7}{3x+11}=1$.

Multiplying by $3x+11$, $9x-7=3x+11$.

Subtracting -7 and $3x$, $6x=18$.

Dividing by 6, $x=3$.

Exercise 113. Monomial and Polynomial Denominators

1. Solve the equation $\frac{x+2}{x-2}+4=7$, and check.

Solve the following equations:

2. $\frac{x+10}{x-3}+\frac{x}{5}=\frac{x(x+3)}{5(x-3)}+\frac{5}{x-3}$.

3. $\frac{10x+17}{18}+\frac{7-5x}{9}=\frac{2(6x+1)}{13x-16}+\frac{1}{3}$.

4. $\frac{2x}{3}-\frac{2x+4}{3}+\frac{5}{9}=\frac{13-7x}{3(2x+1)}-\frac{2}{9}$.

5. $\frac{2x}{5}+\frac{2(1-x)}{7x-6}+\frac{13}{15}=\frac{2x+1}{5}+\frac{2}{5}$.

6. $\frac{9x+5}{14}-\frac{7-8x}{2(3x+1)}+\frac{1}{2}=\frac{36x+15}{56}+\frac{69}{56}$.

7. $\frac{x}{x-1}-\frac{5}{2x-2}-\frac{1}{3}=\frac{8}{3x-3}+\frac{x}{1-x}-\frac{1}{18}$.

Exercise 114. Problems

1. A contractor A has the facilities for doing a piece of work in 4 da., and another contractor, B, can do it in 5 da. Assuming that they can work together at the same rates and that it is desirable to hasten the job by so doing, how many days will it take both together to do the work?

Let x = the number of days it will take both together.

Then $\frac{1}{x}$ = the part that both together can do in 1 da.,

$\frac{1}{4}$ = the part that A alone can do in 1 da.,

$\frac{1}{5}$ = the part that B alone can do in 1 da.,

and $\frac{1}{4} + \frac{1}{5}$ = the part that both together can do in 1 da.

Therefore $\frac{1}{4} + \frac{1}{5} = \frac{1}{x}$.

Solving, $x = 2\frac{2}{9}$.

2. A can do a piece of work in a days and B can do it in b days. How many days will it take them working together? Evaluate the result for $a = 4$, $b = 5$.

3. A can do a piece of work in 4 da., B in 5 da., and C in 6 da. How long will it take them working together?

4. A can do a piece of work in a days, B in b days, and C in c days. How many days will it take them working together? Evaluate the result for $a = 4$, $b = 5$, $c = 6$.

5. A and B together can build a wall in 12 da., A and C in 15 da., B and C in 20 da. In how many days can they build the wall if they all work together?

By working 2 da. each they build $\frac{1}{12} + \frac{1}{15} + \frac{1}{20}$ of it. Why?

If, working together, they require x days, what part of the wall do they build in 2 da.?

6. A and B together can build a wall in 10 da., A and C in 12 da., and B and C in 15 da. In how many days can they build the wall if they all work together?

141. Equations involving Decimals. In practical measurements we are coming more and more to use decimals, and hence we meet decimals in equations. The use of decimals has the advantage that frequently we do not need to clear the equations of fractions in the ordinary sense.

1. Solve $0.7\,x - 0.3 = 0.05\,x + 1$.

Subtracting $0.05\,x$, $\quad 0.65\,x - 0.3 = 1.$

Adding 0.3, $\quad 0.65\,x = 1.3.$

Dividing by 0.65, $\quad x = 2.$

Check. $\quad 0.7 \times 2 - 0.3 = 0.05 \times 2 + 1,$

for $\quad 1.4 - 0.3 = 0.1 + 1 = 1.1.$

2. Solve $0.4\,x + 0.3 - \dfrac{0.4\,x - 1}{x - 0.2} = \dfrac{0.2\,x - 0.1}{0.5}$.

Multiplying by 10 both terms of the two fractions,

$$0.4\,x + 0.3 - \frac{4\,x - 10}{10\,x - 2} = \frac{2\,x - 1}{5}.$$

The second member, $\dfrac{2\,x - 1}{5}$, evidently equals $0.4\,x - 0.2$, and hence

$$0.4\,x + 0.3 - \frac{2\,(2\,x - 5)}{2\,(5\,x - 1)} = 0.4\,x - 0.2.$$

By subtraction, $\quad 0.5 = \dfrac{2\,x - 5}{5\,x - 1}.$

Multiplying by $5\,x - 1$, $\quad 2.5\,x - 0.5 = 2\,x - 5.$

Subtracting -0.5 and $2\,x$, $\quad 0.5\,x = -4.5.$

Dividing by 0.5, $\quad x = -9.$

Check. $\quad -3.6 + 0.3 - \dfrac{-3.6 - 1}{-9 - 0.2} = \dfrac{-1.8 - 0.1}{0.5}$, for each member reduces to -3.8.

3. Solve $0.7\,x + 6.5 = 8 + 0.05\,x$.

Multiplying by 100, $\quad 70\,x + 650 = 800 + 5\,x.$

Subtracting 650 and $5\,x$, $\quad 65\,x = 150.$

Dividing by 65, $\quad x = 2\frac{4}{13}.$

Exercise 115. Equations involving Decimals

Examples 1 to 3, oral

1. Solve $x = 5 + 0.5\,x$; $x = 7 + 0.5\,x$; $x = 9 + 0.5\,x$.
2. Solve $x = 3 + 0.25\,x$; $x = 6 + 0.25\,x$; $x = 9 + 0.25\,x$.
3. Solve $0.2x = 1 - 0.3\,x$; $0.3x = 1 - 0.2\,x$; $0.4x = 1 - 0.1\,x$.

Solve the following equations and check:

4. $0.4\,x + 5 = 9$.
5. $0.7\,x + 3 = 11$.
6. $0.9\,x + 7 = 25$.
7. $0.09\,x + 7 = 25$.
8. $0.09\,x + 11 = 10.8$.
9. $0.5\,x - 6 = 0.2\,x$.
10. $0.7\,x - 5 = 0.2\,x$.
11. $2.5\,x - 24 = 1.3\,x$.
12. $3.9\,x - 4.8 = 2.7\,x$.
13. $5.3\,x - 0.68 = 1.9\,x$.

14. Solve the equation $0.3\,x + 7.2 = 4.8 - 0.02\,x$, (1) by clearing of decimal fractions before subtracting any terms from both members; (2) by first subtracting certain terms from both members and then dividing by the decimal coefficient of x.

Solve the following equations and check:

15. $0.7\,x - 3 + \frac{2}{5}x^2 = 0.3 + 0.4\,x(1 + x)$.

16. $\dfrac{3.48\,x - 8 - 0.4\,x}{8 + 0.01\,x} = 5$.

17. $\dfrac{2.725\,x}{0.1\,x - 18} + 5 = \dfrac{1.635 + 0.1\,x - 18}{0.02\,x - 3.6}$.

18. If 7 is increased by 0.75 of a certain number, the sum is 9.7. What is the number?

19. If 4 is increased by 0.45 of a certain number, the sum is 9.4. What is the number?

20. If 0.4 is added to 0.72 of a certain number, the sum is 22. What is the number?

142. Ratio. The relation of one number to another number of the same kind, as expressed by the indicated division of the first number by the second, is called the *ratio* of the first number to the second.

The ratio of a to b is indicated thus: $\frac{a}{b}$, a/b, or $a : b$.

Hence all ratios may be looked upon as fractions, and the subject of ratio is properly a part of the chapter on fractional equations.

Indeed, it is largely a matter of tradition that we have ratios instead of ordinary fractions in any of our work. We seldom employ the word "ratio" in business, commonly speaking of dividing a sum in the proportion of 2 to 3 instead of dividing it in the ratio of 2 to 3.

143. Terms of a Ratio. The two numbers involved in a ratio are called the *terms* of the ratio. The dividend is called the *antecedent* and the divisor is called the *consequent.*

Thus we have

$$\frac{a}{b} = \frac{\text{antecedent}}{\text{consequent}} = \frac{\text{numerator}}{\text{denominator}} = \frac{\text{dividend}}{\text{divisor}}.$$

If $\frac{a}{b} = \frac{2}{3}$, we say that a is to b as 2 is to 3.

The following important law relating to ratios is frequently used:

Both terms of a ratio may be multiplied by the same number or both terms may be divided by the same number without changing the value of the ratio.

A ratio being a fraction, the laws of fractions hold true for ratios, and it has been seen that we may multiply both terms or divide both terms of a fraction by the same number without changing its value.

For example, the ratio 20 : 25 may be written in simpler form, thus:

$$20 : 25 = \tfrac{2}{2}\tfrac{0}{5} = \tfrac{4}{5}.$$

Similarly, if we wish to express the ratio 7 to 9 so that the consequent shall be 45, we have

$$7 : 9 = \tfrac{7}{9} = \tfrac{3}{4}\tfrac{5}{5} = 35 : 45.$$

In the same manner, $1 : 1 = 2 : 2 = 9 : 9 = \ldots$.

Exercise 116. Ratios

Examples 1 to 4, oral

1. Express in simplest fractional form the ratio of 10 to 20; of 32 to 40; of 45 to 54; of 49 to 56; of 26 to 39.

2. Express in simplest form, as an improper fraction, the ratio of 15 to 10; of 20 to 12; of 35 to 21; of 81 to 63.

3. How is the ratio of 9 to 5 changed by multiplying both terms by 2? by adding 2 to both terms? by subtracting 2 from both terms?

4. How is the ratio of 6 to 9 changed by multiplying both terms by 3? by dividing both terms by 3? by adding 3 to both terms? by subtracting 3 from both terms?

Simplify the following ratios:

5. $a^2 - b^2 : a^2 - 2ab + b^2$. **6.** $a + 1 : a^2 + 2a + 1$.

7. Separate 50 into two parts having the ratio 2 : 3.

Let x = the smaller part.

Then $50 - x$ = the larger part.

Therefore $\frac{x}{50 - x} = \frac{2}{3}$.

Solving, $x = 20$, and $50 - x = 30$.

Separate into two parts having the ratios specified:

8. 24; 1 : 2. **9.** 91; 2 : 5. **10.** 272; 7 : 9.

11. A wheel of diameter 30 in., making 300 R.P.M. (revolutions per minute), is belted to another wheel of diameter 15 in. Find the speed of the smaller wheel.

12. Two cogwheels are geared together and the distance between their centers is 20 in. What are their diameters if their speeds have the ratio of 4 to 5?

144. Use of Ratio in Statistics. We have already seen the value of graphic work in commercial affairs, and particularly in statistics, and are now prepared to see one of the principles upon which its validity rests. This principle is as follows:

A series of numbers may be represented by lines, rectangles, circles, or other geometric magnitudes, provided that the lengths of any two lines representing two of the numbers have the same ratio as the numbers represented; or provided that the areas of any two rectangles representing two of the numbers have the same ratio as the numbers they represent; and so on.

If the numbers to be represented are large and the space on the paper is small, we first determine the length of the longest line that can conveniently be drawn to represent the largest number of the series. Suppose, for example, that its length is 8 cm., and the quantities to be represented, the number of long tons of pig iron produced in the United States annually in recent five-year intervals, to be as follows:

Year	Production
First	9,446,000
Second	13,789,000
Third	22,992,000
Fourth	27,304,000

If a line 8 cm. long represents 27,303,000 T., a line 1 cm. long will represent 27,304,000 T. $\div$ 8, or 3,413,000 T. Now the smallest division easily distinguished on the common scale is the millimeter, or 0.1 cm. Since 1 cm. represents 3,400,000 T. approximately, 1 mm. represents 340,000 T. Dividing the numbers of the above table by this number, we get 27.8, 40.6, 67.6, and 80.3 respectively. Lines of these lengths, in millimeters, have approximately the same ratios as the numbers of tons in the table.

Exercise 117. Ratios in Statistics

Taking any convenient length as a unit, represent graphically the numbers in the following statistics relating to products in the United States:

1. Wheat:

Year	Bushels
1871	300,000,000
1881	449,000,000
1891	476,000,000
1901	522,000,000
1911	635,000,000

2. Pig iron:

Year	Tons
1870	1,800,000
1880	4,300,000
1890	9,000,000
1900	13,800,000
1910	27,300,000

3. Anthracite coal:

Year	Tons
1880	33,000,000
1890	45,000,000
1900	54,000,000
1910	65,000,000

4. Corn:

Year	Bushels
1880	1,600,000,000
1890	1,700,000,000
1900	2,100,000,000
1910	2,900,000,000

5. Cotton:

Year	Bales
1880	6,300,000
1885	6,400,000
1890	8,600,000
1895	7,100,000
1900	10,300,000
1905	10,800,000
1910	12,000,000
1915	11,200,000

6. Gold:

Year	Value
1880	36,000,000
1885	32,000,000
1890	33,000,000
1895	47,000,000
1900	79,000,000
1905	86,000,000
1910	96,000,000
1915	101,000,000

These statistics are, of course, only approximations, but they are substantially correct.

145. Proportion. An expression of equality between two ratios is called a *proportion.*

Thus $\frac{a}{b} = \frac{c}{d}$ is a proportion, and a, b, c, d are said to be in proportion.

This proportion may be written $a : b = c : d$, and is often read, "a is to b as c is to d," or "the ratio a to b is equal to the ratio $c : d$." In older works the above proportion is often written $a : b :: c : d$.

146. Terms of a Proportion. In a proportion the first and fourth terms are called the *extremes*, and the second and third terms the *means.*

In the proportion $a : b = c : d$ the extremes are a and d, and the means are b and c.

147. Mean Proportional. If the means of a proportion are the same, each mean is called the *mean proportional* between the two extremes.

Thus m is the mean proportional between x and y if $x : m = m : y$.

148. Laws of Proportion. The following are two of the important laws of proportion:

1. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $\frac{a}{b} = \frac{c}{d}$.

Multiplying by bd, $ad = bc$.

2. *If the product of two numbers is equal to the product of two other numbers, either two may be made the means of a proportion, and the other two the extremes.*

Let $ad = bc$.

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$,

or $\frac{a}{b} = \frac{c}{d}$.

Exercise 118. Proportion

Examples 1 to 9, oral

1. Solve the equation $\frac{1}{2}x = \frac{2}{3}$.

2. In the proportion $\frac{x}{2} = \frac{2}{3}$ find the value of x.

3. In the proportion $x : 2 = 2 : 3$ find the value of x.

Find the value of x in each of the following proportions:

4. $\frac{x}{4} = \frac{3}{8}$.

5. $\frac{x}{5} = \frac{7}{15}$.

6. $\frac{2}{3} = \frac{x}{9}$.

7. $\frac{3}{5} = \frac{12}{x}$.

8. $\frac{7}{x} = \frac{42}{48}$.

9. $\frac{36}{48} = \frac{3}{x}$.

10. If the means of a proportion are 3 and 12, and one extreme is 4, what is the other extreme?

11. If $x - 5 : 5 = 28 : 10$, what is the value of x?

Find the value of x in each of the following proportions:

12. $x : 2.4 = 1.21 : 1.1$.

13. $x - 12 : 12 = 5 : 1$.

14. $2.6 : x = 1.1 : 7.7$.

15. $17 - x : x = 16 : 18$.

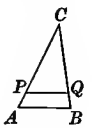
16. If each mean of a proportion is 8, and one extreme is 32, what is the other extreme?

17. In a lever PW, if sufficient power (p) is applied at P, a weight (w) at W can be lifted. If F is the fulcrum, it is known that the ratio $p : w$ is equal to the ratio $FW : FP$. If $FW = 10$ in., $FP = 30$ in., and $w = 240$ lb., find p.

18. If two boys weigh respectively 100 lb. and 120 lb., where must the fulcrum be placed under a 10-foot board so that the boys sitting at the ends will just balance?

The law given in Ex. 17 also applies in the case of this example; that is, we have $100 : 120 = x : 10 - x$.

19. It is proved in geometry that a line parallel to the base of a triangle divides the other two sides proportionally. In this figure, if $AP = 1\frac{1}{2}$, $PC = 6$, and $QC = 5$, find the length of BQ.

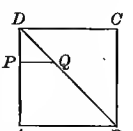


Teachers should use their discretion as to the problems to be assigned on pages 185 and 186. If the students have had no geometry of any kind, most of this work should be omitted.

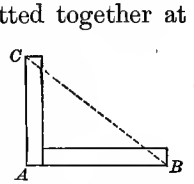
20. It is proved in geometry that two triangles of the same shape have their corresponding sides proportional. In Ex. 19, if $AB = 8$, $AC = 7\frac{1}{2}$, and $PC = 6$, what is the length of PQ?

21. If two sides of a triangle are 7 in. and 9 in., and a line parallel to the third side cuts the first side into the parts 3 in. and 4 in., into what lengths will it cut the second side?

22. In this square, PQ is parallel to AB. If a side of the square is 10 in., $DB = 14.14$ in. If $DP = 3$ in., what is the length of DQ?

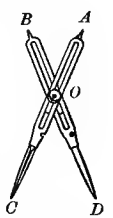


23. Two pieces of timber 1 ft. wide are fitted together at right angles as here shown. AB is 8 ft. long, AC is 6 ft. long, and the distance BC along the dotted line is 10 ft. A carpenter finds it necessary to saw along the dotted line. Find the length of the slanting cut across the upright piece of timber; across the horizontal piece.

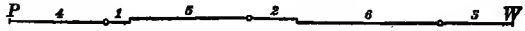


$x : 10 = 1 : 6$ is one proportion.

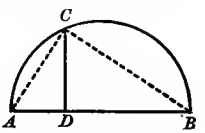
24. This figure represents a pair of proportional compasses used by draftsmen. By adjusting the screw at O, the lengths OA and OC, and the corresponding lengths OB and OD, may be varied proportionally. The triangle formed by O, A, and B is the same shape as that formed by O, C, and D. If $OA = 3$ in. and $OC = 5$ in., then AB is what part of CD?



25. In the following combination of levers a power of 100 lb. is applied at P. What weight can be raised at W?

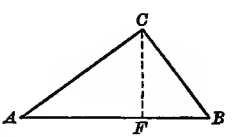


26. It is proved in geometry that if AB is the diameter of the semicircle here shown, CD is the mean proportional between AD and DB. If $AD = 3$ and $CD = 4$, what is the length of DB?



27. In the same figure, if $DB = 4\,AD$ and $CD = 12$, what are the lengths of AD and DB?

28. If a perpendicular is drawn from the vertex of the right angle upon the hypotenuse of a right triangle, it divides the right triangle into two triangles similar to the original triangle and to each other. In the given figure, if $AF = 5$ and $CF = 3.5$, what is the length of FB?



29. In the same figure, if $AB = 7.45$ and $AC = 6.1$, what is the length of AF?

30. If a boy $4\frac{3}{4}$ ft. tall casts a shadow $4\frac{1}{2}$ ft. long at the same time that the school building casts a shadow $67\frac{1}{2}$ ft. long, how high is the school building?

31. If a boy, lying down with his eye to the ground, sights over the top of a 10-foot pole held vertically $6\frac{1}{4}$ ft. from his eye, and can just see the top of a tree $37\frac{1}{2}$ ft. from his eye, what is the height of the tree?

32. If $\dfrac{a+b}{x} = \dfrac{x}{a-b}$, find a formula in x^2.

33. A tree casts a shadow 48 ft. 3 in. long at the same time that a flagstaff 32 ft. high casts a shadow 28 ft. 4 in. long. Find to the nearest inch the height of the tree.

Exercise 119. Review of Chapter X

1. Solve the equations $\frac{2x}{5}+3=\frac{x}{3}+4$ and $\frac{2}{x}+\frac{3}{x}+\frac{4}{x}=1$.

2. Solve for p the equation $\frac{p+pr}{a}+b=p-\frac{p-pr}{c}$.

3. In the study of electricity the formula $C=\frac{en}{R+nr}$ is often found. Solve for n.

4. In Ex. 3, if $C=0.37$, $R=5$, $r=2$, and $e=1.05$, find the value of n.

5. The use of a certain drug in the United States has increased 50% every 5 yr. and at the present time 8000 lb. of the drug are consumed each year. How many pounds of the drug were consumed 15 yr. ago in the United States?

6. The records of two workmen show that A can join the parts of 500 sink mats in 24 hr., and B can do the same in 30 hr. If these are the only available men, what is the smallest number of working days of 8 hr. each in which the employing company can guarantee to finish 2000 sink mats?

7. The total exports of the United States in a certain year were valued as follows: to Europe, \$1,136,000,000; to North America, \$386,000,000; to South America, \$174,000,000; to Asia, \$61,000,000; to Africa, \$19,000,000. Represent these quantities by lines or bars having approximately the same ratios as the quantities themselves.

8. If $a:b=c:d$, show that $a:c=b:d$.

9. If $x:y=p:q$, show that $x+y:y=p+q:q$.

Add 1 to each member of the equation $\frac{x}{y}=\frac{p}{q}$ and reduce each member of the resulting equation to a fraction.

10. If $x:y=p:q$, show that $x-y:y=p-q:q$.

11. If $m:n=x:y$, show that $m+n:x+y=n:y$.

Exercise 120. Review of Chapters I–X

1. How much greater is 17 than 11? $x-1$ than $x-3$? $2x-5$ than $x+6$?

2. Multiply $x^3-3x^2+5x-11$ by $2x-3$, and check the result.

3. Solve $C=\frac{5}{9}(F-32)$ for F. Use the result to change a temperature of 40° C. to Fahrenheit.

4. Multiply $a+b$ by $c+d$, and from the result derive a short, practical method for multiplying $8+\frac{1}{2}$ by $6+\frac{1}{4}$, that is, for multiplying $8\frac{1}{2}$ by $6\frac{1}{4}$.

5. Prove that if x, y, and z represent the three digits of a number $100x+10y+z$, it follows that the number is divisible by 3 if $x+y+z$ is so divisible.

6. The records of a business show gross sales of \$1,250,000 for the year. The cost of doing business for the year was \$200,000 and the net profit was \$300,000. Assuming that the same ratios are maintained, what must be the selling price of articles that cost \$300 to cover the cost of doing business and profit?

7. Derive a one-day rule for interest at $3\frac{3}{4}\%$. Use this rule to find the interest on \$8400 for 24 da. at $3\frac{3}{4}\%$.

8. If stocks of par value \$100 are bought \$1.75 below par and sold \$2.50 above par, how much is made on each share? Represent the process algebraically.

9. Calculate the selling price and profit on an article if the cost of doing business is 18% of the selling price, the profit is 45% of the selling price, and the proportional cost of doing business is \$3.60 for this article.

10. Solve the equation $\dfrac{1}{x-2}-\dfrac{1}{x-3}=\dfrac{1}{x-4}-\dfrac{1}{x-5}$, and check the result.

CHAPTER XI

EQUATIONS APPLIED TO COMMERCE

149. Net Cost. The original cost of goods before freight, commission, cartage, insurance, and other overhead charges (overhead, or burden) are added, is called the *net cost.*

Before considering further commercial problems it is necessary to define a few terms that are in common use in business.

150. Gross Cost. The net cost of goods plus all overhead charges to the time of sale is called the *gross cost.*

151. List Price. The price of goods as given in a printed list or catalogue is called the *list price.*

Retail dealers usually mark their goods instead of issuing a catalogue. The term *marked price* has the same meaning as the term *list price.*

152. Net Price. The price at which goods are actually sold, after all discounts are deducted, is called the *net price.*

153. Profit. The difference between the gross cost and the net selling price of goods is called the *profit.*

If the profit is a negative number it becomes a *loss.*

154. Illustrative Problem. A man loses 15% on the gross cost of his house by selling it for \$6290. What was the gross cost of the house?

If C is the gross cost, the selling price is $C - 0.15\,C$, or $0.85\,C$.

Then since $0.85\,C =$ the selling price,

we have $0.85\,C = 6290$.

Dividing by 0.85, $C = 7400$.

Hence the gross cost of the house was \$7400.

Exercise 121. Commercial Problems

1. I gain 12% on an investment of \$2000. Write a formula for finding the gain and the selling price, and then find each.

2. A man began business Jan. 1, investing \$12,000. On Dec. 31 his resources and liabilities were as follows:

Resources		Liabilities	
Cash on hand	3 500	Accounts payable	1 200
Merchandise	7 200	Notes payable	1 100
Real estate	3 000		
Fixtures	500		
Accounts receivable	4 200		

Find his rate of gain or loss, based on his investment of \$12,000. Write a formula for the general case.

3. The annual report of a dairy company for a recent year shows the following facts relating to the output of butter for each of its four plants:

Plant	Average Cost per Pound	Average Selling Price per Pound	Total Output for Year in Pounds	Annual Profit	Rate of Profit per Pound
A	18¢	25¢	250,000		
B	20	30	180,000		
C	22	35	300,000		
D	21	32	100,000		
Totals					

Find the rate of profit per pound for each plant and enter it in the last column; find the output and annual profit, and from these find the rate of profit on the cost and enter this in the last column. Write a formula for the general case.

In the following table use C for total cost, including overhead charges, r for rate of gain or loss, g for gain, and l for loss, and fill the vacant places:

	Total Cost	Rate of Gain or Loss	Gain or Loss
4.	C	r	
5.	C		l
6.		r	g
7.	\$5200		\$300

8. A dealer sold some goods for S dollars and received r per cent commission. Find the amount of his commission. Find the amount if $S = 725$ and $r = 3\frac{1}{2}$.

9. A merchant bought a bill of goods for G dollars and received a discount of d per cent. Find the net cost of the goods. Find the net cost if $G = 7500$ and $d = 6$.

10. If I gain g dollars on a set of furniture that cost me C dollars, including all overhead, what per cent do I gain? Apply the formula to a problem involving numbers.

11. If I gain c cents on a pair of overshoes that cost me S dollars, including all overhead, what per cent do I gain? Apply the formula to a problem involving numbers.

Notice the difference in denominations in Exs. 10 and 11.

12. If N foot pounds of power are supplied to a machine, and only n foot pounds are used effectively, what is the rate of efficiency of the machine?

The teacher should explain the meaning of "foot pound" if necessary, or refer the pupils to the dictionary.

13. Solve Ex. 12 for $N = 400$, $n = 350$.

155. Trade Discount. It is a general custom that dealers, jobbers, and merchants who buy directly from the manufacturers or wholesalers are allowed a discount from the catalogue or list prices of goods. Such a discount allowed to the trade is called a *trade discount.*

156. Discount Series. If large quantities are ordered, a further discount from the *first net price* is often allowed, then a discount from the *second net price*, and so on, this resulting in a *chain of discounts*, or a *discount series.*

157. Terms. The conditions of payment for a bill of goods are called the *terms*, and these are usually stated on the bill.

Thus a bill on which 60 da. is allowed for payment but which may be *discounted for cash*, say at 2% if paid within 10 da., will have at the top the following: "Terms: 2% 10, net 60," or "Terms: 2/10, N/60."

An allowance made on a bill because it is paid before the date of the bill is called *anticipation.*

A bill of goods may be shipped in July for the holiday trade, simply to relieve the factory, and be accompanied by the bill dated Nov. 1, terms 2/10, N/30. This means that the buyer has a discount of 2% if he pays within 10 da. from Nov. 1, and must pay within 30 da. therefrom. But the buyer may anticipate the payment by sending the money any time before Nov. 1, and then the anticipation will reduce the bill. The usual anticipation is from $\frac{1}{2}\%$ a month to 1% a month.

158. Discount Formulas. If the list price is L and there is a single rate of discount $r\%$,

$$N = L - r\%\ L = L(1 - r\%).$$

If there are two discounts, $r_1\%$ and $r_2\%$, we have

$$N = L(1 - r_1\%)(1 - r_2\%), \text{ and so on.}$$

Since $L(1 - r_1\%)(1 - r_2\%) = L(1 - r_2\%)(1 - r_1\%)$, it makes no difference which discount is taken first.

Discard in the last result any fraction less than $\frac{1}{2}¢$.

Exercise 122. Trade Discount

1. If the list price is L and there is a chain of discounts $r_1\%$, $r_2\%$, $r_3\%$, what is the net price?

2. Apply Ex. 1 when $L = 750$, $r_1 = 6$, $r_2 = 4$, $r_3 = 3$.

3. In Ex. 1 show that the result is the same whatever the order of the discounts.

4. Apply Ex. 3 to the case given in Ex. 2.

5. Find the net price of a book listed at b with discounts of $m\%$, $n\%$.

6. Apply Ex. 5 to the case of a book listed at $80\,¢$ with discounts of 20%, 5%, or, as it is usually written, with discounts of 20, 5.

7. Apply Ex. 1 to a bill for a stove listed at \$30 with the chain of discounts 20, 10, 5.

8. Apply Ex. 1 to a bill for the following: 24 boxes oranges @ \$5.70, discounts 20, 5, 5.

9. The Conover Hardware Company sold the following bill of goods: 2 doz. locks @ \$18 less 20, 10, 5; $1\frac{1}{2}$ doz. locks @ \$20 less 20, 5; 5 doz. $2\frac{1}{2}''$ bolts @ \$1.50 less 30, 5. Find the amount of the bill.

2 doz. locks @ \$18 means that 2 doz. locks have been sold at \$18 per dozen, and similarly for other cases of this kind.

10. The American Woolen Company bought the following: 1500 yd. lining @ \$0.35 less 10, 10, 5; 850 yd. worsted suitings @ \$1.70 less 20, 10, 10; 500 yd. woolens @ \$1.50 less 10, 10. Find the amount of the bill.

11. The National Market Company receives two bids: one offers to supply 1200 lb. butter @ \$0.30 less 20, 10; the other 1200 lb. @ \$0.31 less 10, 10, 10 for the same quality of butter. Which is the more advantageous for the company?

159. Net-Cost-Rate Factor. As explained in § 158, the net price in the case of a chain of two discounts is found by the formula

$$N = L(1 - r_1\%)(1 - r_2\%),$$

and in the case of a chain of three discounts by the formula

$$N = L(1 - r_1\%)(1 - r_2\%)(1 - r_3\%),$$

and so on, whatever the number of discounts in the chain.

But evidently it would be much easier if we had a single number by which to multiply L, instead of multiplying successively by several numbers.

For example, if the list price is \$1000, and the rates of discount are 6%, 5%, we have $1 - 0.06 = 0.94$ and $1 - 0.05 = 0.95$, so that we must multiply \$1000 by 0.94, and the result by 0.95 to find the net price.

It would be easier, however, if we knew the product of 0.94 and 0.95 and could multiply by that. Now this product is 0.893, and, if we knew this, the operation would be very simple.

\$1000
.94
\$940
.95
\$893

The single multiplier which is equal to the product of all the multipliers of the form $1 - r\%$ used in a chain of discounts is called the *net-cost-rate factor.*

That is, in the case of the chain 8%, 6%, 4%, the net-cost-rate factor is $(1 - 0.08)(1 - 0.06)(1 - 0.04)$, or $0.92 \times 0.94 \times 0.96$, which is equal to 0.830208. Using this particular net-cost-rate factor we have to make only three different multiplications, by 8, 3, and 2.

Because of the convenience in using the net-cost-rate factor, salesmen usually prepare tables based on the rates which they ordinarily use in their computations. Such a table is shown on pages 195 and 196.

It would be mathematically simpler to quote a single discount in each case, obviating the necessity for such tables, and the inertia of custom is responsible in large measure for the use of the present plan.

160. Use of Table. The use of the following table of net-cost-rate factors will be understood from a single example. If we are finding the net cost of goods listed at \$85 and subject to the chain discount $12\frac{1}{2}$, 10, 5, we look in the table under $12\frac{1}{2}$, the largest factor, and opposite 10, 5, and find 0.74813 as the net-cost-rate factor. We then multiply \$85 by 0.7481, or practically 0.7481 by 85, only four decimals being necessary when the list price is less than \$100, and we have \$63.59 as the net price.

```
  0.7481
      85
 3 7405
59 848
63.5885
$63.59
```

TABLE OF NET-COST-RATE FACTORS

RATE %	5	10	$12\frac{1}{2}$	$16\frac{2}{3}$	20	25	$33\frac{1}{3}$
Net	95	90	875	83333	80	75	66667
$2\frac{1}{2}$	92625	8775	85313	8125	78	73125	65
5	9025	855	83125	79167	76	7125	63333
5 $2\frac{1}{2}$	87994	83363	81047	77188	741	69469	6175
5 5	85738	81225	78969	75208	722	67688	60167
5 5 $2\frac{1}{2}$	83594	79194	76995	73328	70395	65995	58663
10		81	7875	75	72	675	6
10 $2\frac{1}{2}$		78975	76781	73125	702	65813	585
10 5		7695	74813	7125	684	64125	57
10 5 $2\frac{1}{2}$		75026	72942	69469	6669	62522	55575
10 10		729	70875	675	648	6075	54
10 10 5		69255	67331	64125	6156	57713	513
20 5					608	57	50667
20 10 5					5472	513	456

In the table decimal points are to be inserted as needed.

The table on page 195 is here continued for the cases of 35%, 37½%, 45%, 50%, 66⅔%, 75%, and 80%.

Rate %	35	37½	45	50	66⅔	75	80
Net	65	625	55	50	33333	25	20
2½	63375	60938	53625	4875	325	24375	195
5	6175	59375	5225	475	31667	2375	19
5 2½	60206	57891	50944	46313	30875	23156	18525
5 5	58663	56406	49638	45125	30083	22563	1805
5 5 2½	57196	54996	48397	43997	29331	21998	17599
10	585	5625	495	45	3	225	18
10 2½	57038	54844	48263	43875	2925	21938	1755
10 5	55575	53438	47025	4275	285	21375	171
10 5 2½	54186	52102	45849	41681	27788	20841	16673
10 10	5265	50625	4455	405	27	2025	162
10 10 2½	51334	49359	43437	39488	26325	19744	15795
10 10 5	50018	48094	42323	38475	2565	19238	1539
10 10 5 2½	48767	46891	41264	37513	25009	18757	15005
20	52	5	44	4	26667	2	16
20 5	494	475	418	38	25333	19	152
20 10	468	45	396	36	24	18	144
20 10 5	4446	4275	3762	342	228	171	1368
20 10 10 5	40014	38475	33858	3078	2052	1539	12312

Of course it will be understood that this particular table is merely illustrative. For example, in some lines of business it would be necessary to have the table so arranged as to allow for the discounts 4, 2, or 6, 4, 2, or some other chain that would be used frequently. Such a table is easily made to cover all discounts that a business house ever uses, and this is what is actually done by the bookkeeper or salesman.

Exercise 123. Net-Cost-Rate Factors

1. Using the table on pages 195 and 196, find the net amount of a bill for \$92.50 less 10, 5, $2\frac{1}{2}$.

The chain of discounts is really 10%, 5%, $2\frac{1}{2}$%, but business men have no time for unnecessary symbols.

2. In the table no factor is given under 5 and opposite 10. Why is this? If the factor were given, what would it be? Find the net amount of a bill for \$75 less 5, 10.

3. In the table on page 195 certain spaces are left blank. After finding what spaces should be filled, compute and insert the necessary numbers.

4. In a certain business it is necessary to have the major rates as given at the top of the preceding table, and to have the factors for the minor rates $7\frac{1}{2}$, 5. Compute these factors.

Using the table, find the net amounts of the following bills:

5. 300 bags coffee @ \$30 less $33\frac{1}{3}$, 10, 5; 50 bags coffee @ \$40 less 25, 10, 10; 80 bags coffee @ \$45 less 45, 10, 5.

6. 9 bbl. sugar XXX @ \$12 less 5, $2\frac{1}{2}$; 10 bbl. sugar XX @ \$11 less 5, 5.

7. A dealer remembers that the net price of an article was \$2.70 after a discount of 25, 10 had been allowed, but he cannot find the list price. What was it?

8. A dealer buys galvanized iron pails in lots of 100, thereby being allowed discounts of 45, 20, 5 from the list price of 20¢ each. If he sells them at the list price, what is his rate of profit on the net cost?

9. On a bill of goods listed at \$2875 with discounts 25, 10, 10, 5, what difference does it make in the net price if you take the factor to four decimal places instead of five? How is it with a bill of \$250? with a bill of \$75?

161. Fixing Retail Prices. Until recently, profits were computed as a certain per cent of the cost of the goods. As business became more complicated, the profits were computed on the cost plus the overhead charges, and this is still the common method in small lines of business. When salesmen or agents are employed to sell goods, however, their commission is some per cent of the selling price, and this enters into the overhead charges, overhead, burden, or cost of doing business. In a large establishment, therefore, it is more logical to estimate the cost of the goods, the overhead charges, and the profit, each as a per cent of the selling price.

By keeping careful records of their business, including all the kinds of overhead charges, wholesale dealers are able so to standardize their expenses as to determine what per cent of the selling price should be allowed to overhead charges to realize the expected profit, and *to compute this profit on the selling price instead of on the gross cost.*

A dealer paid \$24.50 for an article, plus 75¢ for cartage. The cost of doing business has been found to be 18% of the selling price, and the profit is to be 10% of the selling price. Find the selling price.

We have $\quad \text{cost} = \$24.50 + \$0.75 = \$25.25.$

Let $\quad x = \text{selling price}.$

Then $\quad 0.18\,x + 0.10\,x = 0.28\,x, \text{ overhead + profit.}$

Hence $\quad x - 0.28\,x = \$25.25,$

or $\quad 0.72\,x = \$25.25;$

whence $\quad x = \$25.25 \div 0.72 = \$35.07.$

From 100% subtract the sum of the rate of profit and the rate allowed to overhead, both based on the selling price as found from experience.

Divide the cost of the goods by this remainder, and the quotient is the selling price.

162. Cofactor. The quotient of 1 divided by any given number is called the *cofactor* (or *reciprocal*) of that number.

We can simplify the solution given on page 198 if we know the cofactor of 0.72, because it is easier to multiply by this cofactor than to divide by 0.72.

That is, it is easier to multiply by 0.04 than to divide by 25, and the result is the same because $x \div 25 = x \cdot \frac{1}{25} = x \cdot \frac{4}{100} = 0.04\,x$.

Similarly, it is much easier to multiply by 0.3183, which is equal to $\frac{1}{3.1416}$, than to divide by 3.1416.

It is therefore desirable to have a table of the cofactors of numbers which we have frequent occasion to use in fixing prices. With such a table the prices may be quickly fixed, and business houses do well to recognize this fact.

The average of overhead charges for a series of years fixes this as a standard in any given establishment, at least for the time being. This and the rate of profit determine the market price. A house which can cut the per cent of overhead charges by efficiency devices and checks can either increase the per cent of profit or undersell its competitors. Hence such a device as the cofactor table, which saves a great deal of time, has a high commercial value.

A single example will illustrate the advantage of using cofactors in ascertaining the selling price.

A house whose average overhead is 17% and rate of profit 12% buys some goods for \$1775. At what price must it sell the goods?

1.4085
1775
7 0425
98 595
985 95
1408 5
2500.0875
\$2500.09

Looking at the table on page 200, in the horizontal line to the right of 17% and under 12%, the cofactor 1.4085 is found. Multiplying \$1775 by 1.4085 we obtain \$2500.09 as the selling price, allowing 29% of it to cover overhead and profit. This is close enough for work with such a table. The house will naturally charge \$2500, which is the exact result.

Selling-Price Cofactor Table

Overhead	Per Cent of Profit Desired							
	1%	2%	3%	4%	5%	6%	7%	8%
15%	1.1905	1.2048	1.2195	1.2346	1.25	1.2658	1.2821	1.2987
16%	1.2048	1.2195	1.2346	1.25	1.2658	1.2821	1.2987	1.3158
17%	1.2195	1.2346	1.25	1.2658	1.2821	1.2987	1.3158	1.3333
18%	1.2346	1.25	1.2658	1.2821	1.2987	1.3158	1.3333	1.3514
19%	1.25	1.2658	1.2821	1.2987	1.3158	1.3333	1.3514	1.3699
20%	1.2658	1.2821	1.2987	1.3158	1.3333	1.3514	1.3699	1.3889
21%	1.2821	1.2987	1.3158	1.3333	1.3514	1.3699	1.3889	1.4085
22%	1.2987	1.3158	1.3333	1.3514	1.3699	1.3889	1.4085	1.4286
23%	1.3158	1.3333	1.3514	1.3699	1.3889	1.4085	1.4286	1.4493
24%	1.3333	1.3514	1.3699	1.3889	1.4085	1.4286	1.4493	1.4706
25%	1.3514	1.3699	1.3889	1.4085	1.4286	1.4493	1.4706	1.4925
	9%	10%	11%	12%	13%	14%	15%	20%
15%	1.3158	1.3333	1.3514	1.3699	1.3889	1.4085	1.4286	1.5385
16%	1.3333	1.3514	1.3699	1.3889	1.4085	1.4286	1.4493	1.5625
17%	1.3514	1.3699	1.3889	1.4085	1.4286	1.4493	1.4706	1.5873
18%	1.3699	1.3889	1.4085	1.4286	1.4493	1.4706	1.4925	1.6129
19%	1.3889	1.4085	1.4286	1.4493	1.4706	1.4925	1.5152	1.6393
20%	1.4085	1.4286	1.4493	1.4706	1.4925	1.5152	1.5385	1.6667
21%	1.4286	1.4493	1.4706	1.4925	1.5152	1.5385	1.5625	1.6949
22%	1.4493	1.4706	1.4925	1.5152	1.5385	1.5625	1.5873	1.7241
23%	1.4706	1.4925	1.5152	1.5385	1.5625	1.5873	1.6129	1.7544
24%	1.4925	1.5152	1.5385	1.5625	1.5873	1.6129	1.6393	1.7857
25%	1.5152	1.5385	1.5625	1.5873	1.6129	1.6393	1.6667	1.8182

Exercise 124. Cofactor Table

1. Calculate for the table on page 200 two additional columns headed 25% and 30% respectively.

2. Using the table, determine the selling price of goods costing \$300 plus \$21 for freight and \$5 for cartage, the overhead being 18% and the profit 15% of the selling price.

3. J. B. Price and Co. buy some wagons for children, listed at \$2.20 each, discounted at 20, 10, with total freight charge of \$8. Their cost of doing business is 15%, and their profit basis is 12%. Using the chain-discount and cofactor tables, find the price at which they must mark each wagon.

Fill out the last column in the following table:

	Cost of Goods	Overhead	Profit	Selling Price
4.	\$245.50	21%	8%	
5.	\$530.75	18%	8%	
6.	\$945	25%	20%	
7.	\$500	20%	15%	
8.	\$475.50	19%	13%	

9. The Conover Company buys wagons costing \$140. Their cost of doing business is on a 12% basis, and they make a profit of 20% on the selling price. At what price must they sell the wagons?

10. The Eureka Shoe Co. handles shoes which cost them \$2.20 a pair. Their cost of doing business is on a $12\frac{1}{2}\%$ basis, and their profit is down to a $2\frac{1}{2}\%$ basis. At what price must they sell the shoes?

163. Commission. From his work in arithmetic the student is more or less familiar with commission. Much new light, however, is thrown on the subject, and in a very practical way, by algebra. Indeed, it is by means of algebra that we obtain a general view of many modern business rules.

A sum charged by one person for transacting business for another is called *commission*, or *brokerage*.

The person for whom the business is transacted is called the *principal*, and the agent is called a *commission merchant*, *broker*, *collector*, or *factor*.

In actual business there is a distinction in these terms, but for our present purposes they may be used interchangeably. The mathematics of the case is the same whichever term is used.

Commission, or brokerage, is charged on the basis of the entire volume of business transacted. It is usually a certain per cent of the cost when a purchase is made, and of the gross proceeds when a sale is made.

In the case of stocks and bonds the brokerage is usually at the rate of \$1.25 on every \$1000 of par value.

Merchandise sent by a principal to his agent is called a *shipment*, but when the agent receives it he speaks of it as a *consignment*, and frequently the agent is called the *consignee* of the merchandise.

An itemized statement of a sale of merchandise, showing the gross proceeds, charges, commission, and similar items, together with the balance to be remitted to the principal, is called an *account sales*. The balance to be remitted is called the *net proceeds*.

An itemized statement showing the prime cost, charges, and gross cost is called an *account purchase*.

Definitions of this kind are not to be learned, but they should be read understandingly and the terms used correctly.

164. Formulas in Commission. From the explanation given on page 202, it is evident that, if P represents the gross proceeds, C the prime cost, c the commission, and r the rate of commission, we have the following:

$$Pr = c, \qquad c \div P = r, \qquad c \div r = P,$$

$$Cr = c, \qquad c \div C = r, \qquad c \div r = C.$$

The student should tell how each of these formulas is obtained, and should understand clearly the significance of each. The formulas in the upper row apply to sales, those in the lower row to purchases.

Exercise 125. Formulas in Commission

Examples 1 to 4, oral

1. Given that $Pr = c$, find c when $P = \$200$, $r = 4\%$.

2. Given that $c \div C = r$, find r when $c = \$7.50$, $C = \$100$; find C when $c = \$120$, $r = 3\%$.

Find the missing numbers in the following:

	Gross Proceeds	Rate of Commission	Commission or Brokerage	Net Proceeds
3.	\$1000	2%		
4.	\$1000		\$20	
5.	\$8500	$2\frac{1}{2}\%$		
6.	\$1200	$16\frac{2}{3}\%$		
7.		2%	\$40	
8.	\$1500			\$1200
9.		5%	\$85	

10. Find the commission and net proceeds in the following:

New York, Aug. 27, 1920

SOLD FOR THE ACCOUNT OF

J. M. Fisher, Yonkers, N. Y.

By J. L. KING CO.

1920							
Aug.	*3*	*50 bbl. sugar*	*8*	*400*			
	12	*60 bbl. sugar*	*8.50*	*510*			
	20	*100 bbl. sugar*	*9.50*	*950*		*1860*	
		CHARGES					
		Freight, $80, Ins., $15		*95*			
		Commission, 5%					
		Net proceeds					

11. Fill the following account sales:

New York, Aug. 8, 1921

SALE OF MERCHANDISE FOR ACCOUNT OF

M. B. Richards, Auburn, N. Y.

By ROBERTS, CARROLL & CO., *4062 E. Canal St.*

Aug.	*4*	*300 bbl. King apples $3.20*					
		75 bbl. Greenings 2.90					
		CHARGES					
		Drayage		*6*	*75*		
		Commission, 2%					
		Remitted herewith					

12. Find the commission and amount charged to account:

ACCOUNT PURCHASE

Grand Rapids, Mich., 8/20, *1921*

PURCHASE ACCOUNT OF

F. S. King, 800 Third Ave., New York

By T. H. DAVIDSON

1921					
July	15	30 D. R. furniture	$80		
	24	10 Parlor suites	200		
Aug.	15	5 Brass beds	50		
		Charges. Cartage		25	
		Commission, 1½%			
		Amt. charged to your acct.			

Make out account sales, inserting dates and names, and find the balance remitted in each case:

13. Sold: 350 bbl. Acme flour @ $6.20. Charges: Freight, $11.20; cartage, $10.50; commission, 4%.

14. Sold: 125 crates eggs @ $6; 2500 lb. cheese @ 12½¢. Charges: Freight and drayage, $21.75; commission, 4%.

15. A clerk receives $10 a week and 1½% commission on goods sold. If he sells $492 worth of goods in a week, how much is his income for the week? Write a formula.

16. A city grocer sends a man out to the residences of his customers every morning to take orders. If the man receives 8% commission on all orders, and if he takes orders to the amount of $9800 a year, how much does he earn? Write a formula by which all problems of this kind can be solved.

165. Interest. As stated in arithmetic, money paid for the use of money is called *interest.* Interest is simply rent paid for the use of money, just as we pay rent for the use of a house or a car.

If the interest is paid in advance on money borrowed at a bank, it is called *discount.*

As already learned in arithmetic, interest is computed as a certain per cent of the money used. Thus, if a man borrows \$500 at 6% for a year, the interest is 6% of \$500, or \$30.

Money loaned on which interest is paid is called the *principal.*

In the example just given, \$500 is the principal.

The rate per cent of the principal paid in interest for one year is called the *rate of interest.*

In the example above, the rate of interest is 6%.

166. Illustrative Problems. Although the subject of interest is, of course, familiar to the student, the following problems will be of service in recalling the methods used.

1. Find the interest on \$840 at 5% for 3 yr.

Since the rate is 5% a year,

Interest for 1 yr. = 5% of \$840 = \$42.

Interest for 3 yr. = 3 × \$42 = \$126.

2. Find the interest on \$750 at $5\frac{1}{2}\%$ for 3 yr. 6 mo.

Since the rate is $5\frac{1}{2}\%$ a year, and 3 yr. 6 mo. = $3\frac{1}{2}$ yr.,

Interest for 1 yr. = $5\frac{1}{2}\%$ of \$750 = \$41.25.

Interest for $3\frac{1}{2}$ yr. = $3\frac{1}{2}$ × \$41.25 = \$144.38.

Therefore, *to find the interest, given the principal, rate, and time, multiply the principal by the rate, and the product by the number that expresses the time in years.*

167. Formulas in Interest. We may express the rule given on page 206 by using initial letters to represent the elements. In this way we have convenient formulas.

Letting the time be expressed in years, we have

Interest = principal × rate × time, or $i = prt$; therefore

$$\text{principal} = \frac{\text{interest}}{\text{rate} \times \text{time}}, \qquad \text{or } p = \frac{i}{rt},$$

$$\text{rate} = \frac{\text{interest}}{\text{principal} \times \text{time}}, \text{ or } r = \frac{i}{pt},$$

$$\text{time} = \frac{\text{interest}}{\text{principal} \times \text{rate}}, \text{ or } t = \frac{i}{pr}.$$

Thus, since the interest on \$200 for 3 yr. at 6% is \$36,

$$i = 3 \times 6\% \text{ of } \$200 = \$36, \qquad r = \frac{\$36}{3 \times \$200} = 6\%,$$

$$p = \frac{\$36}{3 \times 6\%} = \$200, \qquad t = \frac{\$36}{6\% \text{ of } \$200} = 3.$$

It should also be stated at this time that interest is rarely found at present on odd amounts such as \$127.67 or for odd periods of time such as 2 yr. 3 mo. 19 da.

The sum of the principal and interest is called the *amount.*

That is,
$$\begin{aligned} a &= p + i \\ &= p + prt \\ &= p(1 + rt). \end{aligned}$$

Thus, if it is known that $p = \$100$, $r = 6\%$, and $t = 3$ mo. $= \frac{1}{4}$ yr., then $a = \$100\,(1 + \frac{1}{4} \text{ of } 0.06) = \$100\,(1 + 0.01\frac{1}{2}) = \$101.50$.

The principal less the discount is called the *proceeds.*

Thus
$$P = p - d = p - prt = p(1 - rt).$$

This refers to notes due to a bank, the interest being paid in advance and, as stated in § 165, being known as discount. In this formula P represents the proceeds and d the discount.

Exercise 126. Formulas in Interest

By substituting in the appropriate formula, find the principal in each case that will yield the following interest:

1. \$20, in 2 yr. 8 mo., at 6%.
2. \$280, in 3 yr. 4 mo., at 5%.
3. \$9.59, in 3 yr. 6 mo., at 4%.

By substituting in the appropriate formula, find the principal in each case that will amount to the following:

4. \$482.40, in 3 yr. 4 mo., at 6%.
5. \$1151.41, in 2 yr. 4 mo., at $4\frac{1}{2}$%.
6. \$1210.50, in 2 yr. 5 mo., at 5%.

By substituting in the appropriate formula, find the time in each case, given the principal, rate, and interest:

7. \$840, 5%, \$105.
8. \$625, 4%, \$56.25.
9. \$350, 6%, \$49.
10. \$1250, 6%, \$187.50.

By substituting in the appropriate formula, find the rate in each case, given the principal, time, and interest:

11. \$330, 1 yr. 2 mo., \$19.25.
12. \$68.50, 3 yr. 6 mo., \$9.59.
13. How long will it take \$1 to double itself at 5%?
14. What sum must be put at interest at 6% so that it shall amount to \$1000 in 16 yr. 8 mo.?
15. What sum must be put at interest at 5% so that it shall amount to \$1000 in 20 yr.?
16. If money realizes 5% interest, what principal will yield the same yearly income as is received by a laborer who works throughout the year at \$20 per week?

168. Six Per Cent Method of Finding Interest. The method known as the Six Per Cent Method will often be convenient.

Find the interest on \$420 for 5 mo. 10 da. at 6%.

Since 2 mo. = $\frac{1}{6}$ yr., the rate for 2 mo. is $\frac{1}{6}$ of 6%, or 1%.

The interest for 2 mo. is 1% of \$420	=	\$4.20
The interest for 2 mo. more	=	4.20
The interest for 1 mo. more is $\frac{1}{2}$ of \$4.20	=	2.10
The interest for 10 da. more is $\frac{1}{3}$ of \$2.10	=	.70
The interest for 5 mo. 10 da.	=	\$11.20

We see that *the interest at 6% for 60 days is 0.01 of the principal, and for 6 days 0.001 of the principal.*

For other periods of time the interest can be found from the interest for these periods.

The method is sometimes stated as follows:

To find the interest for 600 da. at 6%, move the decimal point in the principal one place to the left; for 60 da., two places to the left; for 6 da., three places to the left.

Since $r = \frac{6}{100}$ and $t = \frac{60}{360} = \frac{1}{6}$, for 60 da., $i = p \cdot \frac{6}{100} \cdot \frac{1}{6} = \frac{1}{100}p$.

After we find the interest at 6%, we easily find the interest for other rates. Thus the interest at 5% is $\frac{5}{6}$ of that at 6%.

Since bank notes usually run for 30 da., 60 da., or 90 da., and since, by simply glancing at the face of the note, we can tell the discount for 60 da. at 6% by the Six Per Cent Method, we can often find the discount on bank notes without using pencil and paper.

Exercise 127. Six Per Cent Method

Examples 1 to 10, oral

Find the interest at 6% on the following:

1. \$800, for 60 da.
2. \$250, for 60 da.
3. \$400, for 60 da.
4. \$800, for 90 da.
5. \$400, for 30 da.
6. \$400, for 15 da.

Find the interest at 5% on the following:

7. \$1200, for 60 da.

8. \$1800, for 60 da.

9. \$3600, for 30 da.

10. \$6000, for 90 da.

11. Find the interest on \$2375 for 60 da. at 6%, first using the Six Per Cent Method, then using the formula, and finally using the ordinary method of finding the interest for 1 yr. and then for the fractional part of a year. Write a statement telling the advantage of the Six Per Cent Method over the other methods.

12. Using the three methods mentioned in Ex. 11, find the interest on \$850 for 90 da. at 6%; on \$1875 for 30 da. at 6%.

13. A note for \$850 is discounted for 60 da. at 6%. Find the discount and the proceeds.

14. A note for \$1275 is discounted for 30 da. at 6%. Find the discount and the proceeds.

15. A note for \$3500 is discounted for 90 da. at 6%. Find the discount and the proceeds.

Find the discounts at 6% on notes for the following amounts:

16. \$3600, for 90 da.

17. \$2250, for 90 da.

18. \$2500, for 30 da.

19. \$4500, for 90 da.

20. \$4600, for 90 da.

21. \$5000, for 30 da.

22. \$750, for 3 mo. 10 da.

23. \$625, for 2 mo. 15 da.

24. \$750, for 1 mo. 20 da.

25. \$250, for 2 mo. 20 da.

26. \$2200, for 3 mo. 15 da.

27. \$4250, for 1 mo. 18 da.

Find the interest at 5% on notes for the following amounts:

28. \$2500, for 60 da.

29. \$3500, for 90 da.

30. \$600, for 1 mo. 10 da.

31. \$850, for 2 mo. 15 da.

169. One-Day Method in Interest. In large city banks the discount rate varies from day to day. The discount clerks derive a short rule for finding the interest for one day at the announced rate and multiply this result by the number of days to find their discount factor.

1. Derive a one-day rule for finding discount at $4\frac{1}{2}\%$.

Since $i = prt$, $r = \frac{9}{200}$, and $t = \frac{1}{360}$, we see that

$$i = p \cdot \frac{9}{200} \cdot \frac{1}{360} = p \cdot \frac{1}{8000}.$$

Hence we have this rule: *To find the interest for 1 da. at $4\frac{1}{2}\%$, point off three places and divide by 8.*

2. Find the interest on \$94,600 for 24 da. at $4\frac{1}{2}\%$.

Since, in this case, 24 da. happens to be 3×8 da., we may simply take $3 \times \$94.60$, which is equal to \$283.80.

3. Extend the rule found in Ex. 1 for 8 da. and 80 da.

For 8 da. the formula evidently becomes

$$i = p \cdot \frac{8}{200} \cdot \frac{8}{360} = p \cdot \frac{1}{1000},$$

and for 80 da. $i = p \cdot \frac{1}{100}$.

Hence we have this rule: *To find the interest at $4\frac{1}{2}\%$ for 8 da., point off three places; for 80 da., point off two places.*

4. Derive a one-day rule for finding discount at 4%.

Evidently $i = p \cdot \frac{4}{100} \cdot \frac{1}{360} = p \cdot \frac{1}{9000}$.

Hence we have this rule: *To find the interest for 1 da. at 4%, point off three places and divide by 9.*

These abridged rules for finding interest are now evident:

For 1 da. at $4\frac{1}{2}\%$, point off three places and divide by 8.
For 8 da. at $4\frac{1}{2}\%$, point off three places.
For 1 da. at 4%, point off three places and divide by 9.
For 9 da. at 4%, point off three places.

Exercise 128. One-Day Method

Find the interest at $4\frac{1}{2}\%$ on the following:

1. \$72,000, 1 da.	**4.** \$65,000, 2 da.	**7.** \$960, 6 da.
2. \$85,400, 1 da.	**5.** \$72,000, 4 da.	**8.** \$7240, 16 da.
3. \$90,000, 1 da.	**6.** \$72,400, 8 da.	**9.** \$45,700, 28 da.

10. Derive a formula for finding interest for 1 da. at 6%, and from this write out a one-day rule.

11. Consider Ex. 10 at 3% and also at 2%.

12. Derive a formula for finding interest for 1 da. at 5%, and write out the rule. Determine whether it is easier to use this rule or to use the Six Per Cent Method and deduct $\frac{1}{6}$.

13. Show how to use the Six Per Cent Method when the time exceeds 1 yr.

14. If y is the number of years, m the number of months, and d the number of days, show that when $r = 6\%$,

$$i = p \times 0.06\,(y + \tfrac{1}{12}\,m + \tfrac{1}{360}\,d)$$
$$= p\,(0.06\,y + 0.005\,m + 0.000\tfrac{1}{6}\,d),$$

and from this formula derive a rule for finding interest at 6% for a given number of years, months, and days.

In solving Exs. 15–20 use the rule obtained in Ex. 14.

Find the interest at 6% on the following:

15. \$5200, 2 yr. 4 mo.	**18.** \$275, 2 yr. 3 mo. 15 da.
16. \$6000, 3 yr. 1 mo.	**19.** \$150, 3 yr. 1 mo. 18 da.
17. \$5500, 1 yr. 3 mo.	**20.** \$350, 1 yr. 3 mo. 10 da.

21. Find the interest on \$950 from 5/20/19 to 11/24/22, at 4%.

Business men commonly write dates as in Ex. 21, where 5/20/19 means May 20, 1919, and 11/24/22 means Nov. 24, 1922.

170. Simple Interest Tables. As an efficiency device a bank uses interest tables except in cases where the Six Per Cent Method or the One-Day Method is easier.

The interest tables most commonly used are based on 360 da. to the year. These tables are generally used by banks in computing interest (discount) on notes payable to them, but banks often use tables based on 365 da. to the year in computing interest paid by them on accounts of their depositors, as stated on page 80. This plan gives the banks an appreciable advantage on the large sums handled by them. The following shows part of a page of an interest table at 6%:

3 Months, 6%										
Total Days	1000	2000	3000	4000	5000	6000	7000	8000	9000	Days over 3 Months
90	15.00	30.00	45.00	60.00	75.00	90.00	105.00	120.00	135.00	0
91	15.17	30.33	45.50	60.67	75.83	91.00	106.17	121.33	136.50	1
92	15.33	30.67	46.00	61.33	76.67	92.00	107.33	122.67	138.00	2
93	15.50	31.00	46.50	62.00	77.50	93.00	108.50	124.00	139.50	3
94	15.67	31.33	47.00	62.67	78.33	94.00	109.67	125.33	141.00	4
95	15.83	31.67	47.50	63.33	79.17	95.00	110.83	126.67	142.50	5
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . . . .

For example, find from the table the interest on \$2750 for 94 da. at 6%.

Interest on \$2000 = \$31.33
Interest on 700 = 10.97
Interest on 50 = .78
Interest on \$2750 = \$43.08

It is not practicable to give, in a book like this, a full interest table. It is better to show part of a real table, as above, than to give a table that is not of the form generally used in large banks.

Exercise 129. Interest Tables

Using the table on page 213, find the interest at 6% on:

1. \$3500, 90 da.
2. \$3650, 90 da.
3. \$4500, 90 da.
4. \$5500, 90 da.
5. \$4250, 90 da.
6. \$8000, 90 da.
7. \$6500, 91 da.
8. \$3700, 92 da.
9. \$5750, 93 da.

10. Make a 6% interest table for 60, 61, 62, and 63 da. similar to the table on page 213.

From the table made in Ex. 10, find the interest at 6% on:

11. \$750, 60 da.
12. \$825, 60 da.
13. \$975, 60 da.
14. \$3700, 60 da.
15. \$4500, 60 da.
16. \$7500, 60 da.
17. \$2750, 61 da.
18. \$3500, 62 da.
19. \$5000, 63 da.

20. Make a 6% interest table for 30, 31, 32, and 33 da. similar to the table on page 213.

From the table made in Ex. 20, find the interest at 6% on:

21. \$250, 30 da.
22. \$750, 30 da.
23. \$825, 30 da.
24. \$2500, 30 da.
25. \$3750, 30 da.
26. \$8500, 30 da.
27. \$4300, 31 da.
28. \$5500, 32 da.
29. \$7500, 33 da.

30. Make a 5% interest table similar to that on page 213.

From the table made in Ex. 30, find the interest at 5% on:

31. \$375, 90 da.
32. \$750, 90 da.
33. \$825, 90 da.
34. \$2000, 90 da.
35. \$2500, 91 da.
36. \$2750, 92 da.
37. \$7250, 90 da.
38. \$4750, 92 da.
39. \$5250, 93 da.

40. A depositor allowed \$1750 to remain in a bank of deposit, without interest, from July 1 to the following Jan. 1. How much interest would he have gained had he transferred it on July 1 to a savings bank paying $3\frac{1}{2}\%$?

171. Table of Difference of Time. Bankers always find the difference in time between two dates by means of tables. The following is one form of the table of difference of time:

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
January .	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March . .	306	337	365	31	61	92	122	153	184	214	245	275
April . . .	275	306	334	365	30	61	91	122	153	183	214	244
May . . .	245	276	304	335	365	31	61	92	123	153	184	214
June . . .	214	245	273	304	334	365	30	61	92	122	153	183
July . . .	184	215	243	274	304	335	365	31	62	92	123	153
August. .	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October .	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

The exact number of days from any day of any month to the corresponding day of any month within a year is found in the table opposite the first month and under the second. For example, from Aug. 7 to Dec. 7 is 122 da.; from Nov. 16 to Apr. 23 is 158 da., since to Apr. 16 it is 151 da., and to Apr. 23 it is 7 da. more, or 158 da. in all.

Exercise 130. Difference of Time

All work oral

From the table state the number of days from:

1. May 7 to Aug. 7.
2. Apr. 6 to Sept. 6.
3. Jan. 5 to Aug. 20.
4. Feb. 7 to Sept. 15.
5. July 4 to Dec. 25.
6. Apr. 15 to Oct. 20.
7. Nov. 16 to July 17.
8. Sept. 20 to Mar. 25.
9. Oct. 21 to Feb. 20.
10. Nov. 25 to Apr. 15.
11. Aug. 27 to Feb. 5.
12. Aug. 19 to Jan. 7.

172. Exact Interest. Since interest at 5% means that 5% of the principal is to be paid for its use for 1 yr., or 365 da., to charge 5% for only 360 da. is to charge more than 5% for a year. Nevertheless, for convenience in computing interest without tables, 360 da. is commonly taken as 1 yr.

For example, required the exact interest on \$1241 from Dec. 21 to Feb. 9 at 5%.

The time, found by the table on page 215, is 50 da. Then the interest is $\frac{50}{365}$ × 5% of \$1241. Canceling,

$$\frac{\overset{10}{\cancel{50}} \times \cancel{5} \times \overset{17}{\cancel{\$1241}}}{\underset{73}{\cancel{365}} \times \underset{2}{\cancel{100}}} = \frac{\$17}{2} = \$8.50.$$

The relation of common to exact interest is discussed on page 217.

Exercise 131. Exact Interest

Find the exact interest on the following:

1. \$4000 for 30 da. at 6%.
2. \$2500 for 60 da. at 5%.
3. \$3250 from Apr. 2 to July 1 at 5%.
4. \$4500 from May 7 to June 6 at 5%.
5. \$4250 from Jan. 6 to Feb. 2 at 6%.
6. \$4600 from Feb. 9 to Apr. 10 at 6%.
7. \$5000 from May 15 to July 14 at 6%.
8. \$37,500 from Aug. 14 to Nov. 12 at 5%.
9. \$45,750 from Sept. 27 to Dec. 26 at 5%.
10. \$75,000 from Dec. 6 to Feb. 4 at 5%.
11. \$125,000 from May 2 to May 24 at $4\frac{1}{2}$%.
12. \$175,000 from June 3 to July 8 at 5%.
13. \$250,000 from Aug. 4 to Aug. 17 at 5%.

173. Exact and Common Interest. Having now considered both exact interest at 365 da. to 1 yr. and common interest at 360 da. to 1 yr., we may discuss algebraically the relation between exact interest and common interest.

The common interest for 1 da. is evidently found by the formula $i = pr \cdot \frac{1}{360}$,

while the formula for exact interest for 1 da., say i', is

$$i' = pr \cdot \tfrac{1}{365}.$$

Dividing the first formula by the second, member by member, we have

$$\frac{i}{i'} = \frac{pr \cdot \frac{1}{360}}{pr \cdot \frac{1}{365}} = \frac{1}{360} \times \frac{365}{1} = \frac{73}{72}.$$

Therefore, by multiplying both sides by i', we have

$$i = 1\tfrac{1}{72}\, i'.$$

That is, *the common interest is equal to the exact interest increased by $\frac{1}{72}$ of itself.*

Similarly, $$i' = \tfrac{71}{73}\, i.$$

That is, *the exact interest is equal to the common interest decreased by $\frac{1}{73}$ of itself.*

Hence to find the exact interest on any sum, we may first find the common interest by any one of the standard methods and then diminish it by $\frac{1}{73}$ of itself.

For example, find the exact interest to be added to taxes which were due not later than Dec. 2 and were paid Dec. 24, the taxes being \$850 and the penalty 10% per annum.

We may consider the penalty the same as interest at 10%.

We then have $i = prt = \$850 \times 0.10 \times \frac{22}{365} = \$5.12$.

If we have only interest tables based on 360 da. to the year, we may find that the common interest is \$5.194, which, diminished by $\frac{1}{73}$ of itself, amounts to \$5.12.

Exercise 132. Exact and Common Interest

Find the difference between the exact interest and the common interest on \$27,500 and on \$125,000, for these times and rates:

1. 60 da., 6%.
2. 90 da., 6%.
3. 30 da., 5%.
4. 2 da., $2\frac{1}{2}$%.
5. 15 da., $4\frac{1}{2}$%.
6. 18 da., $5\frac{1}{2}$%.

7. A man delayed paying his taxes on property assessed at \$92,500, the rate being 0.023, and as a result was required to pay interest on the taxes due for 37 da. at 6%. Find the total amount which the man must pay.

Cities usually calculate all interest charges by the exact method.

8. The sum of daily balances on B's deposit account for July was \$225,400. He is allowed 1 day's exact interest on this total at the rate of 3% per annum. Calculate this interest by finding the common interest by the One-Day Method and then using the formula $i' = \frac{7}{7}\frac{2}{3}\, i$.

9. Our government, in settling a claim for \$3500, paid exact interest for 90 da. at 6%. How much more interest would have been paid if it had allowed common interest?

10. By how much does the common interest exceed the exact interest on \$3000 at $4\frac{1}{2}$% from July 1 to Sept. 10?

11. Give a simple method for changing the common interest on d dollars for 73 m days to the exact interest on d dollars for the same time at the same rate.

12. Explain how the formula $i = \frac{7}{7}\frac{3}{2}\, i'$ or the formula $i = i' + \frac{1}{7}\frac{}{2}\, i'$ makes it easy to calculate the common interest on the same sum and at the same rate for 12 da., 18 da., 24 da., 36 da., 48 da., 72 da., 144 da., and so on, when the exact interest is known.

13. Find the common interest and the exact interest on \$900 for 120 da. at $4\frac{1}{2}$%.

Exercise 133. Review of Chapter XI

1. The Lesher Company bought 2500 yd. of linings, listed at 30¢ per yard, discounts 20, 10, 5. They sold the goods so as to cover overhead of 18% of the selling price, and a profit of 20% of the selling price. Find the selling price of the 2500 yd. and the price per yard.

2. G. K. George discounted his 90-day note for $10,000 at a bank at 4½%. With the proceeds he purchased cloth at $2 per yard, investing all but $7.50. During the 90 da. he disposed of all but 250 yd. of the cloth at an average price of $3.25, depositing the proceeds of the sales in the bank. At the end of the 90 da. the bank took $10,000 from the amount deposited by Mr. George to meet the note. How much was his profit for the 90 da.?

3. If the gross cost of goods is c, the cost of doing business d per cent of the selling price x, and the profit p per cent of x, derive a formula for x.

4. Using the formula $a = p(1 + rt)$, find the value of p and state the rule for finding the principal which will amount to a given sum at a given rate in a given time.

5. Applying the formula derived in Ex. 4, find what principal invested at 6% will amount to $8000.23 in 90 da.

6. Since $a = p + prt$, it is possible to find r when a, p, and t are given. Solve this equation for r.

7. At what rate will $6000 amount to $6067.50 in 3 mo.?

8. Solve for t the equation given in Ex. 6. State the rule for finding the time in which a given principal will amount to a given sum at a given rate.

9. Using the formula derived in Ex. 8, find the time, in this case the fractional part of a year, that it will take $5000 to amount to $5037.50 at 4½%. Reduce the result to months.

Exercise 134. Review of Chapters I–XI

1. If l is the length in feet, w the width in inches, t the thickness in inches, and F the number of board feet in a piece of timber, write the formula for F.

2. Find the value of A_0 in the formula $A_0 = \frac{R}{r}[(1+r)^n - 1]$ when $R = 500$, $r = 0.04$, and $n = 2$.

3. If coffee costs c cents per pound by the bag and loses 5% of its weight in roasting, at what price per pound must it be sold to gain 20% on the cost?

4. In Ex. 3 find the selling price when $c = 24¢$.

5. Find the sum of prt, $p'rt$, $p''rt$, and $p'''rt$.

6. An investor has \$850, \$2500, \$7000, and \$3500 all at interest at 6%. Calculate the total interest he will receive each half year on these investments.

7. Find the sum of pr_1t, pr_2t, pr_3t, and pr_4t.

8. Using the result of Ex. 7 as a formula, find the total interest on the following for 6 mo.: \$5000 at 4%, \$5000 at $4\frac{1}{2}$%, \$5000 at $4\frac{1}{4}$%, and \$5000 at 5%.

9. Explain the algebraic principle involved in multiplying as follows: $82 \times 78 = 6400 - 4 = 6396$.

10. Factor the expressions $p - prt$ and $(1 + r)^2 - 1$.

11. Explain how the distributive law, $a(b - c) = ab - ac$, affords an easy method for multiplying 784,675 by 999.

12. A discounts at a bank at 6% his 60-day note for \$5000 and invests the proceeds in flour listed at \$6 per barrel, less 20, 10. He sells all but 15 bbl. in the 60 da. at \$7.50 per barrel. With what gross profit can he credit himself on the transaction, provided he bought an integral number of barrels and those that remained unsold are inventoried at cost?

CHAPTER XII

SIMULTANEOUS SIMPLE EQUATIONS

174. Indeterminate Equations. If we have one simple equation containing two unknown quantities, we can find either unknown quantity in terms of the other.

For example, if $2x - 3y = 8$, we can see that $x = \frac{3y + 8}{2}$ and that $y = \frac{2x - 8}{3}$, but this does not tell us the value of either.

Indeed, if $y = 1$ we see that $x = 5\frac{1}{2}$; if $y = 2$, $x = 7$; if $y = 4$, $x = 10$, and so on, there being an indefinite number of values of x and y.

An equation which has an indefinite number of roots is called an *indeterminate equation.*

An equation that is not indeterminate is said to be *determinate.*

175. Two Simple Equations. If we have two simple equations containing two unknown quantities, we can usually derive from them a single equation containing only one unknown quantity.

For example, suppose that $x + y = 17$,

and $x - y = 9$.

Adding member for member, $2x = 26$;

whence $x = 13$.

And because $x + y = 17$,

we have $13 + y = 17$;

whence $y = 4$.

176. Simultaneous Equations. Two or more equations that have the same values for the unknown quantities are called *simultaneous equations.*

177. Elimination. The process by which we cause an unknown quantity to disappear in deriving one equation from a system of equations is called *elimination.*

Thus in § 175 we derived the equation $2x = 26$ from the system of equations $x + y = 17$ and $x - y = 9$, thereby eliminating y. We might equally well derive the equation $2y = 8$, thereby eliminating x.

The most common methods of elimination are (1) by addition (or subtraction, as a special case) and (2) by substitution.

178. Elimination by Addition or Subtraction.

1. Solve the system of equations

$$2x + 3y = 27 \quad (1)$$
$$5x - 2y = 1 \quad (2)$$

Multiplying (1) by 2,	$4x + 6y = 54.$
Multiplying (2) by 3,	$15x - 6y = 3.$
Adding,	$19x = 57.$
Dividing by 19,	$x = 3.$
Substituting 3 for x in (1),	$6 + 3y = 27.$
Subtracting 6,	$3y = 21.$
Dividing by 3,	$y = 7.$

Check. Substituting 3 for x, and 7 for y, in (1) and (2), we have

$$6 + 21 = 27, \text{ and } 15 - 14 = 1.$$

Because y was eliminated by adding two equations, member for member, we say that we have eliminated y by *addition.*

2. Solve the system of equations

$$3x + 2y = 23 \quad (1)$$
$$2x + 3y = 27 \quad (2)$$

Multiplying (1) by 2,	$6x + 4y = 46.$	(3)
Multiplying (2) by 3,	$6x + 9y = 81.$	(4)
Subtracting (3) from (4),	$5y = 35.$	
Dividing by 5,	$y = 7.$	
Substituting in (1) or (2),	$x = 3.$	

In this solution we have eliminated x by *subtraction.*

179. Applications and Methods. Let us now consider a practical case in which this work might be used.

A dairy receives an order for 30 gal. of cream, to contain 25% of butter fat. To cream containing 40% of butter fat the manager wishes to add milk containing 5% of butter fat. How many gallons of each should be taken?

We may solve the problem by using only one unknown.

Thus, let x = the number of gallons of cream.

Then $30 - x$ = the number of gallons of milk to be added.

From the conditions of the problem,

$$0.40\,x + 0.05\,(30 - x) = 25\% \text{ of } 30,$$

or $$0.40\,x + 1.5 - 0.05\,x = 7.5;$$

whence $$0.35\,x = 6,$$

and $$x = 17\tfrac{1}{7}.$$

Therefore $$30 - x = 12\tfrac{6}{7}.$$

That is, $17\frac{1}{7}$ gal. of cream and $12\frac{6}{7}$ gal. of milk should be taken.

Instead of this we may, if we choose, use two unknown quantities, and this is often the simpler plan.

Thus, let x = the number of gallons of cream,

and y = the number of gallons of milk.

Then $x + y = 30$, the total number of gallons ordered, (1)

and $0.40\,x + 0.05\,y = 7.5$, the number of gallons of butter fat. (2)

We now have two equations which we may solve as follows:

Multiplying (1) by 0.05,

$$0.05\,x + 0.05\,y = 1.5. \tag{3}$$

Subtracting (3) from (2), $0.35\,x = 6$;

whence $$x = 17\tfrac{1}{7},$$

and $$y = 12\tfrac{6}{7}.$$

In solving the equations produced by complicated commercial problems the second of the above methods, that of simultaneous equations, is usually the easier.

180. Directions for Elimination. From the preceding solutions we see that in the elimination of an unknown quantity by addition or subtraction we proceed as follows:

Multiply both members of the equations by such numbers as will make the coefficients of one of the unknown quantities numerically equal.

If these coefficients have opposite signs, add the equations member for member; if they have the same signs, subtract.

Solve the resulting equation.

Substitute the result thus found in the simpler of the two given equations and solve for the other unknown quantity.

Check the results by substituting in both the given equations.

We commonly speak of multiplying the equation, meaning thereby that we multiply both members of the equation. Similarly, we speak of adding equations and subtracting equations, meaning that we add or subtract the equations member for member.

Exercise 135. Elimination by Addition or Subtraction

Examples 1 to 8, oral

1. Solve for x,
$$x + y = 9$$
$$x - y = 3$$

2. Solve for y,
$$x + 2y = 7$$
$$x + y = 4$$

3. Solve for x,
$$3x + 2y = 5$$
$$5x - 2y = 3$$

4. Solve for m,
$$7m + 3n = 23$$
$$5m + 3n = 19$$

5. Solve for u,
$$2u + v = 14$$
$$3u - v = 11$$

6. Solve for w,
$$2w + 3z = 20$$
$$w - 3z = 1$$

7. Solve for P,
$$3P + Q = 17$$
$$2P - Q = 8$$

8. Solve for F,
$$2F + 7G = 21$$
$$3F - 7G = 14$$

9. When oats are worth \$1.35 a bushel and corn is worth \$1.80 a bushel, how many bushels of each must be taken to make a mixture of 80 bu. worth \$1.68 a bushel?

Here the equations are $x + y = 80$ and $135\,x + 180\,y = 80 \times 168$.

10. Two grades of coffee worth $30\,¢$ and $38\,¢$ a pound respectively are to be combined to make a mixture of 25 lb. worth $35\,¢$ a pound. How many pounds of each kind must be taken?

Such cases are actually used in large establishments, the question of the blend being considered in deciding upon the proportions.

11. A dairy having on hand 20 gal. of milk containing 4% of butter fat, a supply of milk containing 5% of butter fat, and a supply of cream containing 40%, receives an order for 50 gal. of cream to contain 20% of butter fat. The superintendent wishes to dispose of all the 4% milk and to add enough of the 5% milk and 40% cream to make up the 50 gal. ordered. How many gallons of each must he add?

12. Three grades of wheat worth respectively \$2, \$2.10, and \$2.16 a bushel are to be mixed to form 2400 bu. worth \$2.12 a bushel, the number of bushels of \$2.16-wheat being twice that of the \$2.10-wheat. How many bushels of each kind must be taken?

13. An importer of teas wishes to mix two grades of tea costing respectively $40\,¢$ and $50\,¢$ a pound, so as to have 80 lb. which he can sell at $60\,¢$ a pound and still make a profit of 25% on the cost. Assuming that the teas will blend satisfactorily, how many pounds of each must be taken?

14. Having on hand a supply of alcohol 90% pure, and another supply 96% pure, a wholesale dealer wishes to fill an order for 200 gal. which shall be 95% pure. How many gallons of each kind must he take?

181. Elimination by Substitution. It is often convenient to find from one equation the value of one unknown quantity in terms of the other unknown quantity and to substitute this value in the other equation.

1. Solve the system of equations

$$3x + 7y = 22.4 \quad (1)$$
$$x - 5y = 6 \quad (2)$$

From (2), $x = 6 + 5y.$ (3)

Substituting in (1), $3(6 + 5y) + 7y = 22.4,$

or $18 + 15y + 7y = 22.4.$

Subtracting 18, $22y = 4.4.$

Dividing by 22, $y = 0.2.$

Substituting in (3), $x = 7.$

Check. $3 \times 7 + 7 \times 0.2 = 22.4,$

and $7 - 5 \times 0.2 = 6.$

2. Solve the system of equations

$$5x + 2y = 34 \quad (1)$$
$$7x - 3y = 7 \quad (2)$$

From (1), $y = \frac{34 - 5x}{2}.$ (3)

Substituting in (2), $7x - 3 \cdot \frac{34 - 5x}{2} = 7.$

Multiplying by 2, $14x - 102 + 15x = 14.$

Adding 102, $29x = 116.$

Dividing by 29, $x = 4.$

Substituting in (3), $y = \frac{34 - 5 \cdot 4}{2}$

$= 17 - 5 \cdot 2 = 7.$

3. Solve the system of equations

$$x + 3y = 7$$
$$x - 2y = 2$$

We have $x = 7 - 3y$ and $x = 2 + 2y$; hence $7 - 3y = 2 + 2y$, and $y = 1$. Substituting, we have $x = 4$.

182. Directions for Elimination. From the work on page 226 we see that in the elimination of an unknown quantity by substitution we proceed as follows:

From one of the equations find the value of either unknown quantity in terms of the other.

Substitute this value in the other equation and solve.

Exercise 136. Elimination by Substitution

Examples 1 to 6, oral

1. If $x = 3$ and $x + y = 5$, what is the value of y?
2. If $y = 7$ and $x + y = 16$, what is the value of x?
3. If $x = -2$ and $x + y = 1$, what is the value of y?
4. If $y = -2$ and $x + y = 0$, what is the value of x?
5. If $r = 4$ and $r + r' = 9$, what is the value of r'?
6. If $r = 7$ and $r - r' = 5$, what is the value of r'?

Solve the following equations by substitution:

7. $x + 7y = 26$
   $2x + 3y = 19$
8. $x - 3y = -1$
   $3x + 2y = 19$
9. $x + 4y = 35$
   $3x - 2y = 7$
10. $x - 4y = 1$
    $5x + 2y = 49$
11. $x + 5y = 25$
    $7x + 3y = 47$
12. $x - 2y = 2$
    $6x + 5y = 80$
13. $4.5x - 7y = 7$
    $5.5x + 6y = 59.5$
14. $0.8x + 0.3y = 11.3$
    $2x + 3.5y = 14.5$
15. $5m + 7n = 125$
    $7m - n = 13$
16. $5p - 3q = 27$
    $7q - 3p = 15$
17. $19x - y = 18$
    $27x + 4y = 31$
18. $F + 7M = 26$
    $F - 19M = -52$

Exercise 137. Simultaneous Simple Equations

Solve the following equations:

1. $7x + 3y = 23$
   $5x + 3y = 19$
2. $10x + 3y = 75$
   $8x - y = 43$
3. $11x + 21y = 288$
   $16x - 30y = -126$
4. $1.5P + 1.25Q = 19$
   $1.2P + 0.75Q = 13.2$
5. $0.5M - 0.3N = 2$
   $0.3M - 0.4N = 0.1$
6. $x + 3\frac{1}{2}y = 17$
   $3\frac{1}{2}x + y = 37$

Solve the following equations without first clearing of fractions:

7. $\frac{3}{x} + \frac{4}{y} = 3$
   $\frac{9}{x} - \frac{8}{y} = -1$
8. $\frac{15}{x} + \frac{6}{y} = 8$
   $\frac{21}{x} - \frac{12}{y} = 1$
9. $\frac{x}{3} + \frac{y}{4} = 6$
   $\frac{x}{6} - \frac{y}{3} = -2\frac{1}{2}$
10. $\frac{15}{2x} + \frac{1}{y} = 2$
    $\frac{25}{2x} - \frac{3}{y} = 1$
11. $\frac{43.2}{5x} - \frac{6.2}{5y} = \frac{4}{5}$
    $\frac{64.8}{5x} + \frac{12.4}{5y} = 2.6$
12. $\frac{1}{2x} + \frac{1}{y} = p$
    $\frac{1}{2x} - \frac{1}{y} = q$
13. $\frac{a}{x} + \frac{b}{y} = c$
    $\frac{b}{x} + \frac{a}{y} = d$
14. $\frac{1}{ax} + \frac{1}{by} = c$
    $\frac{1}{bx} - \frac{1}{ay} = d$
15. $\frac{a}{bx} + \frac{b}{ay} = a + b$
    $\frac{b}{x} + \frac{a}{y} = a^2 + b^2$
16. $\frac{a}{x} + \frac{b}{y} = c$
    $\frac{a'}{x} + \frac{b'}{y} = c'$

17. If a rectangle were 2 in. longer and 3 in. wider, its area would be increased by 35 sq. in. If the rectangle were 2 in. shorter and 2 in. wider, the area would be unchanged. Find the dimensions.

Let l = the number of inches of length,

and w = the number of inches of width.

Then lw = the number of square inches of area.

Then $(l + 2)(w + 3) = lw + 35$,

and $(l - 2)(w + 2) = lw$.

Simplifying, $3l + 2w = 29$,

and $l - w = 2$.

Solving, $l = 6.6$,

and $w = 4.6$.

Therefore the dimensions are 6.6 in. and 4.6 in.

Here the initial letters of length and width have been used to represent the unknown quantities, a custom that is coming into use. It is permissible to use x and y, or l and w, or any convenient letters.

18. If a rectangle were 5 in. shorter and 1 in. wider, its area would be decreased by 35 sq. in. If it were 5 in. longer and 1 in. wider, its area would be increased by 65 sq. in. Find the dimensions.

19. The perimeter of a rug is 20 ft., and three times the length plus five times the width is 36 ft. Find the area.

20. The length of a room is $33\frac{1}{3}\%$ greater than the width, and the perimeter is 70 ft. Find the dimensions.

21. The width of a room is two thirds the length, and the length exceeds the width by 7 ft. Find the dimensions.

22. The length of a rectangle is 50% greater than the width, and the perimeter is 160 ft. Find the dimensions.

23. The altitude of a rectangle is 20% less than the base, and the perimeter is 18 in. Find the dimensions.

183. Clearing of Fractions. If the equations contain fractions, it is usually better to clear of fractions before eliminating one of the unknown quantities.

Solve the equations

$$\frac{x+2}{x-2}=\frac{y+8}{y+4} \tag{1}$$

$$\frac{2x-5}{4y-1}=\frac{x-1}{2y+1} \tag{2}$$

Clearing (1), $xy+2y+4x+8=xy-2y+8x-16.$ (3)

Clearing (2), $4xy-10y+2x-5=4xy-4y-x+1.$ (4)

Simplifying (3), $x-y=6.$

Simplifying (4), $x-2y=2.$

Solving, $x=10,$

and $y=4.$

Check. $\frac{10+2}{10-2}=\frac{4+8}{4+4}=\frac{12}{8}$; $\frac{20-5}{16-1}=\frac{10-1}{8+1}=\frac{1}{1}$.

Exercise 138. Clearing of Fractions

Examples 1 to 5, oral

1. If $\frac{1}{x}=3$, what is the value of x?

2. If $\frac{3}{x}=7$, what is the value of x?

3. Find the value of x when $\frac{1}{x}=\frac{1}{7}$; when $\frac{2}{x}=\frac{3}{7}$.

4. If $\frac{1}{x}+\frac{1}{y}=4$ and $\frac{1}{x}-\frac{1}{y}=2$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? of y? Check the results.

5. If $\frac{1}{x}+\frac{2}{y}=4$ and $\frac{1}{x}-\frac{2}{y}=0$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? of y? Check the results.

Solve the following equations:

6. $\frac{x}{3}+\frac{y}{4}=4$

$\frac{x}{4}+\frac{y}{3}=\frac{25}{6}$

7. $\frac{x+y}{3}=\frac{x-y}{5}$

$\frac{x-3y}{7}=2x+7$

8. $\frac{x-4}{x-3}=\frac{y+4}{y+7}$

$\frac{x+5}{x+2}=\frac{y-1}{y-2}$

9. $\frac{x+y}{8}+\frac{x-y}{6}=5$

$\frac{x+y}{4}-\frac{x-y}{3}=10$

10. $\frac{5}{x+2y}=\frac{7}{2x+y}$

$\frac{7}{3x-2}=\frac{5}{6-y}$

11. $\frac{3x-2}{5x-1}=\frac{3y+7}{5y+16}$

$\frac{3x-1}{x+5}=\frac{6y-5}{2y+3}$

12. $\frac{4x+5y}{40}=x-y$

$\frac{2x-y}{3}=\frac{1-4y}{2}$

13. $\frac{x+y}{7}+\frac{x-y}{2}=1$

$\frac{x+y}{2}-\frac{x-y}{8}=7\frac{1}{4}$

14. $\frac{x+2y+1}{2x-y+1}=2$

$\frac{3x-y+1}{x-y+3}=5$

15. $\frac{x+y}{3}+\frac{y-x}{2}=9$

$\frac{x}{2}+\frac{x+y}{9}=5$

16. $7x-\frac{y}{5}=48$

$\frac{35y+x}{7}=26$

17. $\frac{x+y-1}{x-y+1}=7$

$\frac{y-x+1}{x-y+1}=35$

18. $\frac{x+y+3}{x-y-3}=-\frac{3}{2}$

$\frac{x-y-3}{x-y+3}=-2$

19. $\frac{x-y}{x+y}=-\frac{8}{15}$

$\frac{63x-3y-44}{7}=100$

Exercise 139. Miscellaneous Problems

1. When weighed in water, tin loses 0.137 of its weight, and copper 0.112 of its weight. If a 10-pound mass of tin and copper loses 1.195 lb., find the weight of the tin and the weight of the copper in the mass.

Let t = the number of pounds of tin,

and c = the number of pounds of copper.

Then $t + c = 10,$ (1)

and $0.137\,t + 0.112\,c = 1.195.$ (2)

Multiplying (1) by 0.112,

$0.112\,t + 0.112\,c = 1.12.$ (3)

Subtracting (3) from (2), $0.025\,t = 0.075.$

Dividing by 0.025, $t = 3.$

Substituting in (1), $c = 7.$

Therefore there are 3 lb. of tin and 7 lb. of copper in the mass.

2. When weighed in water, silver loses 0.095 of its weight, and copper 0.112 of its weight. If a 12-pound mass of silver and copper loses 1.174 lb., find the weight of the silver and the weight of the copper in the mass.

3. When weighed in water, gold loses 0.051 of its weight, and silver 0.095 of its weight. If a 6-ounce piece of gold and silver loses 0.35 oz., find the weight of the gold and the weight of the silver in the piece.

4. When weighed in water, tin loses 0.137 of its weight, and lead 0.089 of its weight. If a 65-pound mass of tin and lead loses 6.025 lb., find the weight of the tin and the weight of the lead in the mass.

5. An iron bar covered with brass weighs 13 lb. When weighed in water, iron loses 0.128 of its weight, and brass 0.119 of its weight. If the bar loses 1.655 lb. when weighed in water, find the weight of the brass that covers the iron.

6. Twenty-seven coins, dollars and quarters, amount to \$19.50. How many are there of each kind?

Let d = the number of dollars,

and q = the number of quarters.

Then $d + q = 27,$

and $d + \frac{q}{4} = 19.50.$

Explain the second equation and solve.

7. A man distributed \$5.25 in dimes and quarters among thirty boys, each boy receiving a coin. How many boys received dimes? How many received quarters?

8. The charge for admission to an entertainment was 50¢ for adults and 25¢ for children. If the proceeds from 125 tickets were \$51.25, how many tickets for adults and how many tickets for children were bought?

9. A school gave an entertainment for which the tickets were sold to pupils at 40¢ each, and to others at 50¢ each. There were 245 tickets sold and the receipts were \$108.50. How many of each kind were sold?

10. A grocer has in his cash drawer 106 bills, some one-dollar and the rest two-dollar bills. The total amount is \$138. How many bills has he of each kind?

11. A receiving teller at a bank took in 515 bills, some five-dollar and the rest two-dollar bills. The total amount was \$2155. How many of each kind did he receive?

12. A paymaster at a shop has 210 silver coins, some quarters and the rest half dollars. The total amount is \$75. How many coins has he of each kind?

13. A dealer has 19 pieces of iron pipe, some 12 ft. long and the rest 6 ft. long. If the total length of the pipe is 168 ft., how many pieces has he of each length?

184. Three Simultaneous Equations. If three or more simultaneous equations are given which involve three or more unknown quantities, these quantities are eliminated by combining pairs of equations, as shown in the following solution:

Solve the equations

$$5x + 2y - 4z = -3 \quad (1)$$
$$3x - 3y + 5z = 12 \quad (2)$$
$$4x + 5y + 2z = 20 \quad (3)$$

Using (1) and (2), we may eliminate z as follows:

Multiplying (1) by 5, $25x + 10y - 20z = -15.$ (4)

Multiplying (2) by 4, $12x - 12y + 20z = 48.$ (5)

Adding (4) and (5), $37x - 2y = 33.$ (6)

We may eliminate z between (1) and (3), as follows:

Multiplying (3) by 2, $8x + 10y + 4z = 40.$ (7)

Adding (1) and (7), $13x + 12y = 37.$ (8)

We now have two equations, (6) and (8), involving x and y. We may eliminate y as follows:

Multiplying (6) by 6, $222x - 12y = 198.$ (9)

Adding (8) and (9), $235x = 235.$

Dividing by 235, $x = 1.$

Substituting in (8), $13 + 12y = 37.$

Hence $12y = 24,$

and $y = 2.$

Substituting $x = 1$, $y = 2$, in (1), $5 + 4 - 4z = -3.$

Solving for z, $z = 3.$

Therefore $x = 1, y = 2, z = 3.$

Check. Substituting in (1), (2), and (3), we have

$$5 + 4 - 12 = -3,$$
$$3 - 6 + 15 = 12,$$
$$4 + 10 + 6 = 20.$$

Exercise 140. Three Simultaneous Equations

Examples 1 to 4, oral

1. In the system of equations

$$x+y+z=6 \quad (1)$$
$$x+y-z=4 \quad (2)$$
$$x-y+z=2 \quad (3)$$

eliminate y and z at the same time between (2) and (3). What is the value of x?

2. In the equations of Ex. 1 eliminate z between (1) and (2), and find the value of $x+y$. Then substitute the value of x found in Ex. 1, and find the value of y.

3. In the equations of Ex. 1 substitute in (1) the values of x and y found in Exs. 1 and 2, and find the value of z.

4. In the equations of Ex. 1 check the values of x, y, and z found in Exs. 1–3, by substituting in each equation.

Solve the following equations:

5. $x+y+z=10$
$x-y+z=2$
$x+y-z=8$

6. $x+y+z=7$
$3x+y-z=3$
$2x+4y+z=12$

7. $x+y+z=13$
$3x+y-3z=5$
$x-2y+4z=10$

8. $x+2y+3z=41$
$x-3y+4z=6$
$5x+6y-7z=63$

9. $3x+2y-4z=15$
$5x-3y+2z=60$
$2x+4y-3z=45$

10. $x-3y+z=10$
$2x-7y-5z=-2$
$x+y-2z=5$

11. $10x+8y-9z=10$
$12x+2y-16z=15$
$2x+10y-25z=0$

12. $x+y+z=14.6$
$x-y+z=3.4$
$x+y-z=12.2$

13. A farmer who had kept accounts for his farm as a whole, but not for the separate crops, desired to find the average profit per acre of each crop. His records show the following acreage and profits:

	Corn	Wheat	Oats	Profit
First year:	80 A.	20 A.	20 A.	$2400
Second year:	70 A.	30 A.	20 A.	$2500
Third year:	60 A.	30 A.	30 A.	$2300

Find the average profit per acre for each crop.

14. A company announces in its circular that it makes three standard grades of fertilizers containing respectively the following ingredients and selling at the prices stated:

	Nitrogen	Potash	Phosphate	Price per Ton
Grade I:	50 lb.	110 lb.	150 lb.	$23.10
Grade II:	70 lb.	100 lb.	150 lb.	$25.50
Grade III:	90 lb.	90 lb.	120 lb.	$26.10

The expert at an experiment station advises a farmer to use a fertilizer containing 80 lb. nitrogen, 80 lb. potash, and 160 lb. phosphate. How much should he pay the company for a ton of fertilizer made up on this formula?

15. A railroad in its annual reports for three different years, during which there was no great change in operating costs and no change in rates, gave the following statistics:

	Number of Passengers Carried	Number of Tons of Freight	Number of Tons of Express	Net Operating Revenue
First year:	12,400,000	9,600,000	500,000	$4,500,000
Second year:	12,100,000	9,100,000	600,000	4,900,000
Third year:	13,100,000	10,200,000	600,000	6,000,000

Find to the nearest mill the average net operating revenue per passenger, per ton of freight, and per ton of express.

185. Standard Dietaries. In preparing scientific dietaries, based upon the composition of foods and the amount of each ingredient needed by the average person of a certain class, the simultaneous equation is useful. For this purpose the following table of composition, in per cents, and of fuel values (calories) is given, with the recent average cost per pound.

The per cents are given to the nearest tenth; the calories are per pound.

Food Material	Protein	Fat	Carbohydrates	Mineral Matter	Calories	Cost
Beef						
Loin	16.1	17.5		0.9	1025	$0.26
Ribs	13.9	21.2		0.7	1135	.24
Round	19.0	12.8		1.0	890	.25
Dried	26.4	6.9		8.9	790	.30
Veal						
Leg	15.5	7.9		0.9	625	.18
Loin	16.6	9.0		0.9	685	.25
Breast	15.4	11.0		0.8	745	.16
Mutton						
Leg	15.1	14.7		0.8	890	.18
Loin	13.5	28.3		0.7	1415	.25
Pork						
Loin	13.4	24.2		0.8	1245	.22
Ham, smoked	14.2	33.4		4.2	1635	.30
Bacon	9.1	62.2		4.1	2715	.24
Chicken	13.7	12.3		0.7	765	.22
Turkey	16.1	18.4		0.8	1060	.30
Eggs	13.1	9.3		0.9	635	.55
Cod, fresh	11.1	0.2		0.8	220	.13
Mackerel, fresh	10.2	4.2		0.7	370	.14
Salmon, tinned	21.8	12.1		2.6	915	.18
Oysters	6.0	1.3	3.3	1.1	225	.30
Butter	1.0	85.0		3.0	3410	.44
Cheese	25.9	33.7	2.4	3.8	1885	.25

Food Material	Protein	Fat	Carbo-hydrates	Mineral Matter	Calories	Cost
Milk, whole . . .	3.3	4.0	5.0	0.7	310	$0.09
Oatmeal	16.7	7.3	66.2	2.1	1800	.14
Corn meal . . .	9.2	1.9	75.4	1.0	1635	.04
Rice	8.0	0.3	79.0	0.4	1620	.08
Wheat flour, white	11.4	1.0	75.1	0.5	1635	.03
Wheat, b'kf'st foods	12.1	1.8	75.2	1.3	1680	.15
Crackers	9.7	12.1	69.7	1.7	1925	.12
Macaroni	13.4	0.9	74.1	1.3	1645	.14
Sugar			100.0		1750	.05
Beans, dried . . .	22.5	1.8	59.6	3.5	1520	.08
Peas, dried . . .	24.6	1.0	62.0	2.9	1565	.10
Beets	1.3	0.1	7.7	0.9	160	.05
Cabbage	1.4	0.2	4.8	0.9	115	.05
Potatoes	1.8	0.1	14.7	0.8	295	.02
Tomatoes . . .	0.9	0.4	3.9	0.5	100	
Apples	0.3	0.3	10.8	0.3	190	.03
Bananas	0.8	0.4	14.3	0.6	260	.03
Grapes	1.0	1.2	14.4	0.4	295	.12
Oranges	0.6	0.1	8.5	0.4	150	.15
Strawberries . . .	0.9	0.6	7.0	0.6	150	
Almonds	11.5	30.2	9.5	1.1	1515	.20

The following table shows the average amount required per adult person per day:

	Protein	Fat	Carbo-hydrates	Fuel Value
	Ounces	*Ounces*	*Ounces*	*Calories*
Man at hard work	4.70	3.35	15.41	3270
Man at moderate work . . .	3.84	1.87	17.11	2965
Woman at moderate work . .	3.25	1.24	12.	2700

These standards are average findings of several investigations. Other standards differ somewhat. To obtain the amount of each required for a child, take half the amount required by a man at moderate work.

186. Illustrative Example. Determine the quantity of pork loin and the quantity of beans that will supply the protein and fat necessary for one day for a family consisting of a man and woman at moderate work and a child.

By the table, the needs for the family for 1 da. are :

	Protein	Fat	Carbohydrates	Calories
Man	3.84 oz.	1.87 oz.	17.11 oz.	2965
Woman . . .	3.25 oz.	1.24 oz.	12. oz.	2700
Child	1.92 oz.	0.94 oz.	8.56 oz.	1483
	9.01 oz.	4.05 oz.	37.67 oz.	7148

Let x = the number of ounces of pork,

and y = the number of ounces of beans.

Then, from the nearest per cent in the table,

$$0.13\,x + 0.23\,y = 9, \tag{1}$$

and

$$0.24\,x + 0.02\,y = 4; \tag{2}$$

or, multiplying (1) by 100 and (2) by 50 to avoid decimals,

$$13\,x + 23\,y = 900, \tag{3}$$

and

$$12\,x + y = 200. \tag{4}$$

Multiplying (4) by 23,

$$276\,x + 23\,y = 4600. \tag{5}$$

Subtracting (3) from (5),

$$263\,x = 3700;$$

whence

$$x = 14.1, \text{ approximately.}$$

Therefore

$$y = 30.8, \text{ approximately.}$$

Hence the required amount of protein and fat can be secured from 14.1 oz. of pork and 30.8 oz. of beans.

Thus far, however, the problem is not complete, since it does not determine the carbohydrates and the calories. The 14.1 oz. of pork and 30.8 oz. of beans would supply together, as the table shows, 18.4 oz. of carbohydrates and 4023 calories, showing a lack of 19.3 oz. of carbohydrates and 3125 calories. This is considered on the next page.

Of course the question of digestibility is also to be considered.

187. Full Dietary. The example on page 239 did not consider, as already stated, the carbohydrates and calories. These may be provided by bread and sugar, thus:

		Protein	Fat	Carbohydrates	Calories	Cost
Pork	14.1 oz.	1.89 oz.	3.41 oz.		1097	$0.194
Beans	30.8 oz.	6.93 oz.	0.55 oz.	18.36 oz.	2926	.154
Bread	16. oz.	1.82 oz.	0.16 oz.	12.02 oz.	1635	.03
Butter	4. oz.	0.04 oz.	3.40 oz.		853	.11
Sugar	7. oz.			7. oz.	766	.022
		10.68 oz.	7.52 oz.	37.38 oz.	7277	$0.51

The above work does not involve any algebra; it is only a fair approximation and is added for completeness.

Exercise 141. Dietaries

Find the quantity of each of the following foods needed to supply a ration containing 9 oz. protein and 4 oz. fat:

1. Veal loin, beans; also cheese, macaroni.
2. Leg of mutton, beans.
3. Bacon, white bread made of wheat flour.
4. Mutton loin, bread as in Ex. 3.
5. Eggs, breakfast food made of wheat.
6. Taking the results of Exs. 1–5, determine the cost of each ration and arrange the rations in order of expense.
7. Using three unknowns, determine the quantity of oatmeal, milk, and eggs that will supply the necessary protein, fat, and carbohydrates for the breakfast of a man at hard work, assuming that this is to be one fourth of his day's nourishment, and find whether this will supply the necessary 817 calories for a quarter of the day's work.

188. Ratio of Solvency. Two firms, A and B, failed at the same time, their accounts showing the following:

	Assets		Liabilities	
A	Due from B,	$200,000	Owed to B,	$300,000
	From other sources,	700,000	To other parties,	900,000
B	Due from A,	$300,000	Owed to A,	$200,000
	From other sources,	800,000	To other parties,	1,300,000

The court decided that it would be unfair to the other creditors if A and B settled their balance of $100,000 as if they were solvent, but that all assets and liabilities of each must be counted in fixing the *ratio of solvency.*

Let $x =$ the ratio of solvency of A,

and $y =$ the ratio of solvency of B.

Then $\dfrac{\text{Assets of A}}{\text{Liabilities of A}} = \dfrac{200{,}000\,y + 700{,}000}{1{,}200{,}000} = x$, A's ratio,

and $\dfrac{\text{Assets of B}}{\text{Liabilities of B}} = \dfrac{300{,}000\,x + 800{,}000}{1{,}500{,}000} = y$, B's ratio.

Solving, we have $x = \frac{121}{174}$ and $y = \frac{39}{58}$. We now have:

	Assets		Liabilities	
A	$\frac{39}{58} \times \$200{,}000 =$	$134,482.76	$\frac{121}{174} \times \$300{,}000 =$	$208,620.69
		700,000.00	$\frac{121}{174} \times$ 900,000 =	625,862.07
	Total .	$834,482.76	Total . . .	$834,482.76
B	$\frac{121}{174} \times \$300{,}000 =$	$208,620.69	$\frac{39}{58} \times \$200{,}000 =$	$134,482.76
		800,000.00	$\frac{39}{58} \times$ 1,300,000 =	874,137.93
	Total . . .	$1,008,620.69	Total . .	$1,008,620.69

Hence A's assignees pay B's $208,620.69 less $134,482.76, or $74,137.93; A's other creditors are paid $625,862.07, and B's other creditors are paid $874,137.93.

Exercise 142. Ratio of Solvency

Determine the ratio of solvency and method of settlement of the accounts in each of the following cases:

1.	Assets		Liabilities	
A	Due from B,	$400,000	To B,	$300,000
	Other sources,	950,000	To others,	1,400,000
B	Due from A,	$300,000	To A,	$400,000
	Other sources,	800,000	To others,	1,100,000

2.	Assets		Liabilities	
A	Due from B,	$120,000	To B,	$180,000
	Other sources,	500,000	To others,	720,000
B	Due from A,	$180,000	To A,	$120,000
	Other sources,	400,000	To others,	500,000

3.	Assets		Liabilities	
A	Due from B,	$35,000	To B,	$25,000
	Due from C,	42,000	To C,	50,000
	Other sources,	120,000	To others,	180,000
B	Due from A,	$25,000	To A,	$35,000
	Due from C,	45,000	To C,	60,000
	Other sources,	80,000	To others,	85,000
C	Due from A,	$50,000	To A,	$42,000
	Due from B,	60,000	To B,	45,000
	Other sources,	130,000	To others,	200,000

Exercise 143. Review of Chapter XII

1. Solve the simultaneous equations $5x - 3y = 1.9$ and $4x - 11y = -0.2$, checking the results.

2. Solve the equations $mx + ny = a$ and $px + qy = b$.

3. Using the results in Ex. 2 as formulas, substitute the values of m, n, p, q, a, and b as given in the equations in Ex. 1, and thus check the results found in Ex. 1.

4. A confectioner made a profit of \$750 in January by selling 12,000 lb. of candy and 1000 qt. of ice cream. In February his profit on 11,500 lb. of candy and 1200 qt. of ice cream was \$755. What was his profit per unit on each?

5. Solve the equations $ax + y = b$ and $3x - cy = d$, using the method of substitution.

6. A dealer shows two different samples of spring wheat, selling at \$2 and \$2.15 respectively. A buyer orders 1200 bu. at \$2.10 per bushel, to be made up of the two kinds. How many bushels of each kind should be taken?

7. A dairy having on hand 15 gal. of milk containing 4% of butter fat, some milk containing 5% of butter fat, and some cream containing 40% of butter fat, received an order for 60 gal. of cream containing 25% of butter fat. The manager wishes to use the 15 gal. of 4% milk, and enough of the 5% milk and 40% cream to produce the 60 gal. of cream ordered. How many gallons of each should he use?

8. Solve the simultaneous equations $4x - y = 11$, $3y - z = 2$, and $5x + 2z = 17$, checking the results.

9. Determine the quantities of round steak, potatoes, and bread made from wheat flour that will be sufficient to give the necessary protein, fat, and carbohydrates for a day to two men at hard work and two women at moderate work.

Exercise 144. Review of Chapters I–XII

1. Using m to represent the cost of raw material, l the cost of labor, and o the miscellaneous overhead charges in producing n phonographs, write the formula by which to find the value of c, the average cost of a phonograph.

2. Solve the equation $\frac{1}{2}(5x-3)=4x-7$, and check the result.

3. Using the formula $i=prt$, derive a one-day rule for interest at $2\frac{1}{4}\%$.

4. If the cost of goods is c, the cost of doing business b per cent of the selling price, and the profit p per cent of the selling price, find the selling price.

5. Multiply $2x^2-3x+5$ by $5x^2+6x$, and check.

6. Simplify $\frac{a}{b}\div\frac{c}{d}$, and use the result as a formula by which to write the result of $\frac{3}{4}\div\frac{2}{3}$.

7. If I buy goods for \$800, and my overhead charges are 18% of the cost and my profit is 10% of the cost, find the selling price. Find the selling price if the overhead charges and the profit are 18% of the selling price and 10% of the selling price respectively.

8. Using the formula $a=p(1+rt)$, find the number of years in which a principal p doubles itself at r per cent simple interest.

9. Using the result obtained in Ex. 8, find in how many years a principal will double itself at 6% simple interest.

10. Solve the equations $3x-y=5$ and $2x+3y=7$.

11. Using the formula $a=p(1+rt)$, find the rate at which a principal p will double itself in t years at simple interest.

CHAPTER XIII

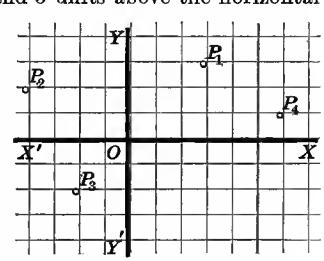
GRAPHS

189. Location of Points on a Map. Points are located on a map by means of latitude and longitude. Latitude is stated in degrees north or south of the equator, and longitude is stated in degrees east or west of the prime meridian through Greenwich, England.

Thus, to the nearest degree, the position of New York City is 41° N., 74° W.; that is, the latitude is 41° north of the equator and the longitude is 74° west of Greenwich.

190. Location of Points on Paper. In a similar way we may locate points on paper. We may take two perpendicular lines, one vertical and the other horizontal, and measure distances to the right and left of the vertical line, and upwards and downwards from the horizontal line.

In this figure P_1 is 3 units to the right of the vertical line YY', which is called the *y-axis* or *axis of y*, and 3 units above the horizontal line XX', which is called the *x-axis* or *axis of x*; P_2 is 4 units to the left of YY' and 2 units above XX'; P_3 is 2 units to the left of YY' and 2 units below XX'; and P_4 is 6 units to the right of YY' and 1 unit above XX'.

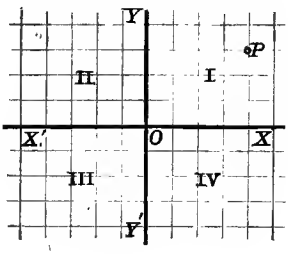


In the same way we may locate a chair on the floor with reference to the east and north walls, a spring in a field with reference to two fences meeting at right angles, or a point on the blackboard with reference to two lines perpendicular to each other.

The students have already studied some of this work in Chapter III.

191. Coordinates. The distances of a point to the right or left of the axis of y and above or below the axis of x are called the *coordinates* of the point.

Thus the point P is in the *first quadrant* (I). It is designated as the point (4, 3), and its coordinates are 4 and 3. A point (− 4, 3) would be represented in the *second quadrant* (II), and so on, distances measured to the right of YY' or above XX' being positive, and distances measured to the left of YY' or below XX' being negative. The coordinates of O, which is called the *origin*, are 0 and 0.



192. Abscissa. The distance of a point from the vertical axis, measured on the axis of x or parallel to that axis, is called the *abscissa* of the point.

Thus the abscissa of P is 4. The abscissa of a point in the second quadrant is negative. The abscissa of a point in the third quadrant is also negative. The abscissa of a point in the fourth quadrant is positive.

193. Ordinate. The distance of a point from the horizontal axis, measured on the axis of y or parallel to that axis, is called the *ordinate* of the point.

The ordinate of P is 3. The ordinate of a point in the first or second quadrant is positive; in the third or fourth quadrant is negative.

194. Plotting a Point. Representing a point by means of its coordinates is called *plotting* the point.

To plot the point (− 2, − 3), take the abscissa − 2 and draw its ordinate − 3. The point is, therefore, in the third quadrant.

195. Coordinate Paper. Paper ruled in squares for convenience in plotting points is called *coordinate paper.*

Coordinate paper, also known as *cross-ruled paper, cross-section paper,* or *squared paper,* will be found of use in the representation of equations and in drawing many of the figures used in geometry, as has already been suggested in Chapter III.

Exercise 145. Plotting Points

Examples 1 to 6, oral

1. In what quadrant is the point (2, 5)?

2. In what quadrants are the points (1, 1), (− 2, − 6), (4, − 2), and (− 2, 4)?

3. Where is the point (0, 0)? (5, 0)? (0, 5)? (− 5, 0)?

4. What is the distance from (4, 0) to (− 4, 0)?

5. In what quadrant is (2, 7)? (− 2, 7)? (2, − 7)?

6. What is the distance from (4, 3) to (− 4, 3)? from (4, 3) to (4, − 3)? from (0, 0) to (0, 7)? from (− 2, 5) to (− 2, − 5)? from (2, − 5) to (− 2, − 5)?

7. Plot the points (1, 3), (7, 6), (5, 2), (9, 1), (1, 9), (5, 5), (7, 0), (0, 7), (0, 0).

8. Plot the points (− 2, 3), (− 4, 6), (− 6, 4), (− 5, 2).

9. Plot the points (− 2, − 3), (− 4, − 6), (− 5, − 2), (− 6, − 9), (− 7, − 7).

10. Plot the points (2, − 3), (3, − 2), (4, − 6), (6, − 4), (7, −7), (− 7, 9), (0, − 4), $(1\frac{1}{2}, 2\frac{1}{4})$, (− 0.5, − 3).

11. Plot the points (4, 4), (2, 1), (4, 1), (− 2, − 4), (− 4, − 4), (− 2, −1), (− 4, −1), (− 2, − 4).

12. Plot the points (3, 5), (2, 3), (1, 5), (2, 6), (5, 3), (2, 0), (− 3, − 5), (− 2, − 3), (−1, − 5).

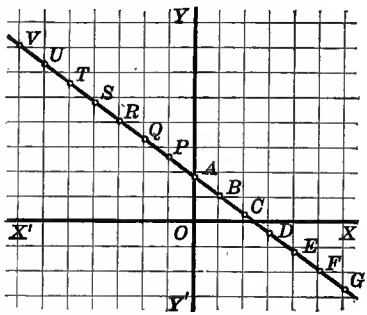
13. Plot the points (1, 4), (4, 5), (4, − 4), (1, − 1), (1, − 2), (1, 3), (− 4, − 5), (−1, −1), (−1, 4), (0, 4).

14. Join (0, 0) and (3, 5), (3, 5) and (6, 0), (1.2, 2) and (4.8, 2), (7, 0) and (12, 5), (7, 5) and (12, 0), (13, 0) and (13, 5), (18, 4) and (17, 5), (17, 5) and (15, 5), (15, 5) and (14, 4), (14, 4) and (14, 3), (14, 3) and (18, 2), (18, 2) and (18, 1), (18, 1) and (17, 0), (17, 0) and (15, 0), (15, 0) and (14, 1). What word is spelled by these lines?

196. Plotting an Equation. Not only is it easy to plot a point, but it is also possible to *plot* or *graph an equation.*

For example, consider the equation $3x + 4y = 7$.

Solving for y we have $y = \frac{7 - 3x}{4}$. That is, y is a function of x, and therefore for any value that we may give to x we may find a corresponding value of y.



Thus, if $x = 1$,

then $y = \frac{7 - 3 \times 1}{4} = 1.$

We may now plot the point (1, 1) and then we have one point, B. In a similar way we may plot several other points as follows:

Let $x =$	0	1	2	3	4	5	6
Then $y =$	$1\frac{3}{4}$	1	$\frac{1}{4}$	$-\frac{1}{2}$	$-1\frac{1}{4}$	-2	$-2\frac{3}{4}$
Point =	A	B	C	D	E	F	G

Let $x =$	-1	-2	-3	-4	-5	-6	-7
Then $y =$	$2\frac{1}{2}$	$3\frac{1}{4}$	4	$4\frac{3}{4}$	$5\frac{1}{2}$	$6\frac{1}{4}$	7
Point =	P	Q	R	S	T	U	V

We might also take fractional values of x and find the corresponding values of y and the corresponding points. *Only two points are necessary, however, to determine the straight line.*

197. Equation of the First Degree. An equation of the first degree in two unknown quantities represents a straight line.

198. Graph. The line that represents an equation is called the *graph* of the equation.

Exercise 146. Graphs of Equations

Examples 1 to 3, oral

1. In the equation $y = 4 - x$ find the value of y when $x = 1$; when $x = 2$; when $x = 4$; when $x = 7$; when $x = 0$; when $x = -2$; when $x = -4$.

2. In the equation $x = y - 5$ find the value of x when $y = 0$; when $y = 1$; when $y = 4$; when $y = 5$; when $y = 10$; when $y = -1$; when $y = -3$.

3. In the equation $x - y = 7$ what is the value of x when $y = 7$? What is the value of y when $x = 7$?

4. Make a table like that on page 248, giving values of y corresponding to six values of x, in the equation $5x - 2y = 7$.

5. Make a similar table for the equation $3x - 4y = 9$.

6. Make a similar table for the equation $2x = 3y$.

7. Plot the equation given in Ex. 4.

8. Plot the equation given in Ex. 5.

9. Plot the equation given in Ex. 6.

Plot the following equations, fixing two points in each:

10. $x + 2y = 5$.

11. $x - 2y = 5$.

12. $3x + 5y = 7$.

13. $3x - 5y = 7$.

14. $5x + 7y = 12$.

15. $7x = 8y$.

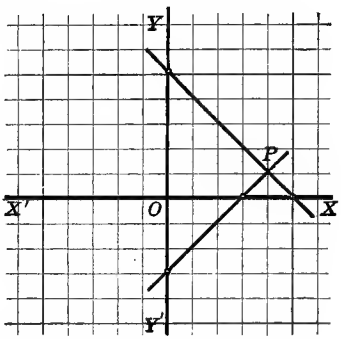
16. $2.7x + 1.2y = 3.9$.

17. $4.1x - 2.3y = 1.9$.

18. Plot the equation the graph of which passes through the points (0, 0) and (5, 5). At what angle does it seem to cut the axis of x?

19. Plot the equation the graph of which passes through the points (0, 0) and (− 5, 5). At what angle does it seem to cut the axis of y?

199. Two Linear Equations. Two linear equations which involve two unknowns are represented by two straight lines. These two straight lines can intersect in only one point. Therefore, in general,



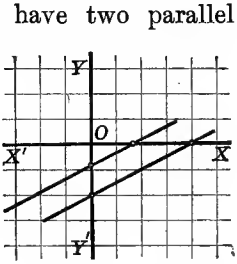
The graphs of two linear equations involving two unknowns have only one point in common.

Two linear equations involving two unknowns have only one pair of values of the unknowns in common.

Thus the graphs of the equations $x + y = 5$ and $x - y = 3$ intersect at P. The coordinates of P are 4 and 1. Hence the solution of the given equations is $x = 4$, $y = 1$.

There are an infinite number of points on each graph, but there is only one point on both graphs.

200. Inconsistent Equations. If we plot the two equations $x - 2y = 4$ and $3x - 6y = 5$, we shall have two parallel lines. Such lines have no point in common. Considering the equations, we see that the second one reduces to $x - 2y = 1\frac{2}{3}$. The equations are therefore *inconsistent*, since $x - 2y$ cannot be equal to both 4 and $1\frac{2}{3}$.



201. Equivalent Equations. If we plot the equations $2x + 4y = 5$ and $x + 2y = 2\frac{1}{2}$, we shall find that one graph coincides with the other. They have an infinite number of points in common, and therefore an infinite number of values of x and y satisfy the equations, every pair of roots of either being a pair of roots of the other. The equations are therefore *equivalent*.

Exercise 147. Graphs of Linear Equations

Examples 1 to 4, oral

1. What is the nature of the graph of $x = 2$? of $y = 2$?

2. What is the nature of the graph of $y = 4$? of $y = -4$?

3. State two points on the graph of $x + 4y = 8$.

4. State two points through which passes the graph of the equation $x = y$.

Plot the following equations, and solve each by measuring the coordinates of the point of intersection of the graphs:

5. $x + 4y = 11$
$2x - y = 4$

6. $2x + 3y = 19$
$7x - 2y = 4$

7. $x + 5y = -3$
$2x - 3y = 20$

8. $2x - 9y = 23$
$5x + y = -13$

9. $x + 5y = 0$
$3x + 9y = -6$

10. $7x + 2y = 14$
$5x - 3y = -21$

11. $2x - 3y = 7$
$5x - 7y = 14$

12. $6x - 3y = 15$
$2x + 7y = 45$

13. Show by graphs that the equations $x + 4y = 6$ and $0.5x + 2y = 4$ are inconsistent.

14. Show by graphs that the equations $0.2x - 0.5y = 6$ and $x - 2.5y = 30$ are equivalent.

15. Show by graphs that the three equations $x + y = 6$, $2x - y = 0$, and $5x + 3y = 22$ have a common root. From the graphs find the root.

16. Show by graphs that the three equations $x + y = 5$, $2x - 3y = 20$, and $3x + y = 2$ have no root common to all.

17. If $x + 5y = 2.1$, y is equal to what function of x? x is equal to what function of y? If $2x - y = 2$, y is equal to what function of x? x is equal to what function of y?

202. Wage Table. The graph will often be found to serve a very important purpose in relation to prices and wages.

For example, a contractor who pays a number of workmen at the rate of 37¢ an hour finds that he saves a great deal of time by using either a wage table or a wage graph. Each method is, of course, based on the formula

$$w = \$0.37\ h,$$

where w is the wages due, and h the number of hours.

The wage table in this case, for the $5\frac{1}{2}$ da. of 8 hr. each in the working week, is as follows:

Hours	0 da.	1 da.	2 da.	3 da.	4 da.	5 da.
0		$2.96	$5.92	$8.88	$11.84	$14.80
1	$0.37	3.33	6.29	9.25	12.21	15.17
2	.74	3.70	6.66	9.62	12.58	15.54
3	1.11	4.07	7.03	9.99	12.95	15.91
4	1.48	4.44	7.40	10.36	13.32	16.28
5	1.85	4.81	7.77	10.73	13.69	¼ hr. $0.09
6	2.22	5.18	8.14	11.10	14.06	½ hr. .19
7	2.59	5.55	8.51	11.47	14.43	¾ hr. .28

Find the wages due a man who worked 4 da. of 8 hr. each, and $3\frac{1}{2}$ hr. the next day.

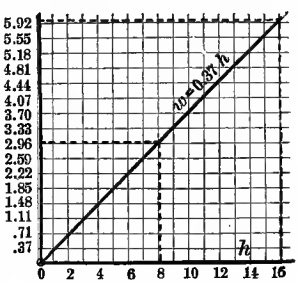
Looking in the table under 4 da. and opposite 3 hr. we find that the wages for 4 da. 3 hr. are $12.95. Looking for ½ hr. in the lower right-hand corner, we find $0.19. Hence the total wages due are $13.14.

The wage table can be used with about the same facility as the wage graph described on page 253.

The wage tables used by paymasters are often in book form, a page or two being devoted to a table at a given rate. The size of the entire table depends largely on the number of different rates included.

It is evident that when time and overtime are paid for at different rates, two different tables must be consulted.

203. Wage Graph. If we draw the graph of the equation $w = \$0.37\,h$, we shall find it to be a straight line, as stated on page 248. On account of the size, we give here only a portion of this graph, leaving it to the student to complete as directed in the exercise below.



From the graph it will be seen that the wages for 8 hr. are \$2.96, as shown by the dotted lines. Similarly, by laying a ruler on the page, or by following the coordinate lines, we see that the wages for 6 hr. are \$2.22, for 12 hr. they are \$4.44, for 14 hr. they are \$5.18, and so on. A little practice makes it very easy to use the wage graph.

Exercise 148. Wage Graphs

1. Complete the above wage graph up to 44 hr.

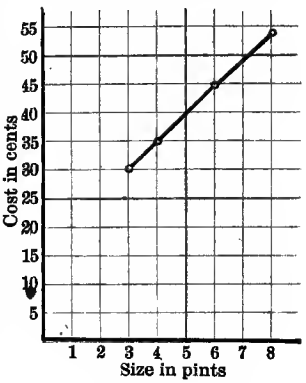
2. From the completed graph of Ex. 1, fill in the following section of a pay roll, the wage per hour being \$0.37:

Men's Numbers	Hours	Amount Due	
23	$18\frac{1}{2}$		
24	33		
27	$37\frac{1}{2}$		
28	40		
29	42		
32	43		
33	44		

3. Draw a wage graph to 44 hr., the wages being 39¢ an hour; that is, draw a graph of the equation $w = 0.39\,h$.

4. Draw a wage graph to 44 hr., the wages being $42\frac{1}{2}$¢.

204. Commercial Interpolation. In commercial calculations many of the numbers vary directly, as in the case of wages and hours; that is, as we double the number of regular hours we double the wages, the wage graph being a straight line. Many other cases occur in which the cost varies directly or approximately as some other factor, such as length, weight, horse power, or capacity. If the graph is a straight line or approximately so, the cost of intermediate sizes or number of articles can be determined very closely by graphic *interpolation*; that is, by finding corresponding intermediate points on the graph.



To illustrate this, consider the case of a house which catalogued aluminum saucepans at the following prices: 3 pt., 30¢; 4 pt., 35¢; 6 pt., 45¢; 8 pt., 54¢, with intermediate sizes at corresponding prices. The house received an order for 18 doz. 5-pint pans and 24 doz. 7-pint pans. How much should it charge for the lot?

In the graph the four points represent the given prices, and they are nearly in the same straight line. We therefore see at a glance that the 5-pint size should cost 40¢ and the 7-pint size 49¢.

Exercise 149. Commercial Interpolation

1. Assuming that for a short period of years the increase in insurance rates varies directly as the increase in age, find the premium per $1000 at ages 22, 23, and 24 years, if at 21 yr. the premium is $17.90 and at 25 yr. it is $19.63.

In this example it is easier to take $17.90 at the origin.

2. The same company as in Ex. 1 charges \$25.88 at 35 yr. and \$30.55 at 40 yr. Find by graphic interpolation the rates for the four intermediate years.

In Ex. 2 the results will not be as close as in Ex. 1.

3. The cost of a certain type of engine for various horse powers (H.P.) is listed as follows: 3 H.P., \$400; 6 H.P., \$600; 12 H.P., \$800. Plot the three points representing these facts, join them by a smooth curve, and determine the approximate cost of engines of H.P. 4, 5, 7, 8, 9, 10, and 11.

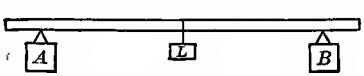
4. A contractor finds that the average expenditure for a group of 50 workmen employed on a job was \$630 per week, while another group of 75 averaged \$945. All the workmen being of the same grade, find from a graph the wages for a week of 19 men; of 32 men; of 68 men; of 96 men; of 125 men; of 132 men.

5. In an experiment a wooden rod was deflected as follows:

Load in grams	50	60	70	80
Deflection in centimeters	1.3	1.5	1.8	2.1

Drawing a graph, find if the deflection varies uniformly with the load, and find the deflection for intermediate loads.

6. A yellow pine rod 30 in. long rests on two supports, A and B, 24 in. apart. Under loads applied at the mid-point, L, it is deflected as follows:



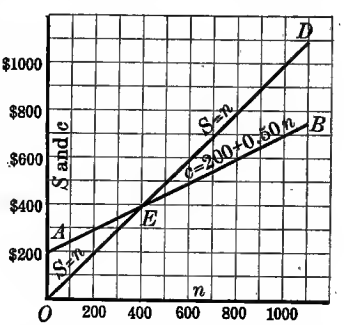
Load in pounds	8	16	21.5	38	49	60.5	70
Deflection in inches	0.14	0.3	0.41	0.77	1	1.35	1.8

Draw the graph and determine the deflection under a load of 10 lb.; of 18 lb.; of 25 lb.; of 42 lb.; of 65 lb.

205. Net-Profit Graph. The net-profit graph is composed of the graphs of the equations of total cost and of total sales. To illustrate, consider the following:

The cost of the necessary machinery and its installation before a certain toy could be manufactured was \$200. Each toy produced costs 50¢ for material and labor and is sold for \$1. The cost equation is $c = 200 + 0.50\,n$, and the sales-receipts equation is $S = n$. Drawing both graphs on one diagram and to one scale, we have the figure shown below, the straight line AB being the graph of $c = 200 + 0.50\,n$ and the straight line OD that of $S = n$.

When $n = 800$, it is easily seen from the diagram that $c = \$600$ and $S = \$800$. It is also seen from the figure that g, the net profit, is \$200 or that portion of the ordinate drawn through 800 which lies between the graphs AB and OD. When $n = 1000$, we easily see from the diagram that $c = \$700$, $S = \$1000$, and $g = \$300$.



In general, if we represent the total receipts from the sale of an article by S, the number sold by n, and the price per unit by p, we have

$$S = np.$$

In this equation the value of n is supposed to be known, and the graph of the equation is a straight line. If k represents the initial cost before any of the units are sold, and k' the cost per unit which must be added to the initial cost, the cost of producing n units is $c = k + k'n$. This is also a linear equation. By subtracting the value of c for any number n from the corresponding value of S, we evidently find the profit made in selling the n articles.

Exercise 150. Net-Profit and Other Graphs

Using the diagram on page 256, consider Exs. 1–4:

1. How many toys must be sold before the company will "break even"?

2. What is the total cost of producing 200 toys? the sales receipts for 200 toys? the loss on 200 toys?

3. How many toys must be sold before the gain is \$100?

4. What is the gain on 1000 toys?

5. If the cost of running a certain store for selling automobile tires averages \$15 per day and the profit is \$1.50 per tire, graph the equation expressing the gain or loss per day if n is the number of tires sold in a day.

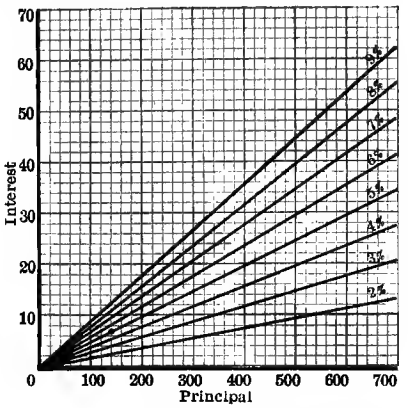
6. The telegraph rate between cities A and B is $20\,¢$ for any number of words up to 10, and $1\,¢$ for each additional word, the equation being $c = 20 + (n - 10)$. Construct the graph of this equation, beginning with $n = 10$.

7. If the first cost of producing a book is \$700 for plates and printing, and each copy represents an additional outlay of $60\,¢$, write the equation of cost. If the book sells at \$1.50, write the equation of sales receipts. Graph these equations, and answer questions analogous to those in Exs. 1–4.

8. The cost of a mine and the installation of machinery was \$125,000. The overhead charges in getting a ton of ore to the smelter were \$1.50 per ton. If the ore was sold at the smelter at an average price of \$3.50 a ton, construct the net-profit graph for this mining venture.

9. From the graph constructed in Ex. 8 find how many tons must be sold to break even; what the net profit or net loss will be when 15,000 T. have been marketed; how many tons must be sold to make a net profit of \$300,000.

206. Interest Graphs. If we take the equation $i = pr$, which gives the interest for 1 yr., and give to r the values 2%, 3%, ···, 9%, we shall have eight linear equations. The graphs of all these equations are shown in the accompanying figure and are sufficient for the purpose of checking computations in interest. Suppose, for example, we wish to find the interest on \$50 for 1 yr. at 3%.



Look above the point 500, which in this example may be taken to represent \$50, on the axis of principals and the ordinate strikes the 3% graph at 15. That is, the interest on \$50 for 1 yr. at 3% is \$1.50.

Of course this graph is too small for practical purposes. If drawn to a scale four times the one here used, and if extended to include other principals, it can be used to advantage, particularly as a check.

It should be noticed that the numbers on the axis of principals may be read as 1, 10, 100, ···; 2, 20, 200, ···, as circumstances demand; also that the experience and intelligence of the student will enable him to determine whether the number read on the interest axis is 0.15, 1.5, 15, or 150, and so on.

Graphs are very valuable as checking devices, but they cannot be used in general unless the one using them has mastered the relations expressed by the algebraic equation of the graph or has had experience in dealing with the numbers whose relation is represented graphically. No device or method of calculating can eliminate the necessity for mastering the mathematics involved in commercial transactions.

Exercise 151. Interest Graphs

1. Draw the graphs on page 258, and continue the principals to 1200 and the interests to 100, extending the interest graphs accordingly.

From the graphs in Ex. 1 find the interest for 1 yr.:

2. On \$75 at 6%.

3. On \$50 at 4%.

4. On \$85 at 3%.

5. On \$62 at 5%.

6. On \$105 at 3%.

7. On \$108 at 4%.

8. On \$110 at 5%.

9. On \$120 at 6%.

Of course, in general, these results are merely approximate. The graph serves, however, as a ready check on more exact work.

10. Draw interest graphs as above for 1 yr. for the following rates: $2\frac{1}{2}\%$, $3\frac{1}{2}\%$, $4\frac{1}{2}\%$, $5\frac{1}{2}\%$.

11. Reversing the use of the graph as given on page 258, find the principal which will yield \$3.60 interest in 1 yr. at 2%; at 3%; at 4%.

12. By interpolating between the graphs find the approximate interest on \$450 for 1 yr. at $4\frac{1}{2}\%$.

13. By interpolating between the graphs find the approximate rate that will yield \$4 interest on \$90 for 1 yr.

14. By interpolating between the graphs find the approximate interest on \$750 for 1 yr. at $4\frac{1}{2}\%$.

15. By interpolating between the graphs find the approximate interest on \$1150 for 1 yr. at $5\frac{1}{2}\%$.

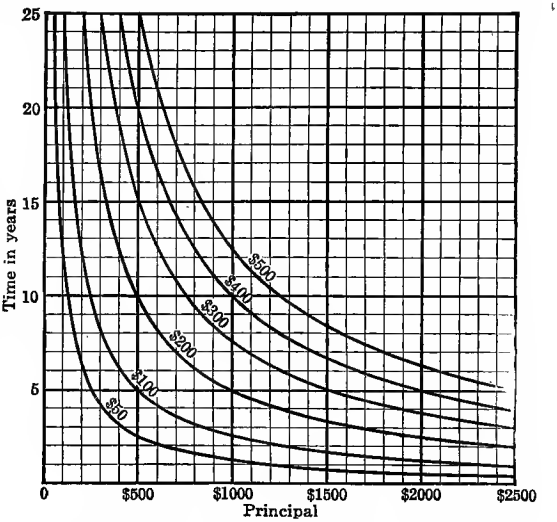
16. Reversing the use of the graphs drawn in Ex. 10, find the principal that will yield \$85 interest in 1 yr. at $3\frac{1}{2}\%$; at $4\frac{1}{2}\%$; at $5\frac{1}{2}\%$.

17. Draw the interest graph for 1 yr. represented by the equation $i = 0.08\,p$.

207. Interest Graphs. Another problem in interest can be quickly solved to a fair degree of approximation by means of graphs of the size shown below. This figure represents a 4% interest graph, the equation $i = prt$ becoming $i = 0.04\,pt$.

The curves are found by locating various points when $i = 50$, then when $i = 100$, and so on.

1. Find the time necessary for \$400 to earn \$100 interest at 4%.



Start at the point for \$400 on the x-axis, or axis of principals; run up the ordinate to the \$100 interest graph, and notice that this is reached $6\frac{1}{4}$ spaces up. That is, the time is approximately $6\frac{1}{4}$ yr.

2. Find the principal which in 10 yr. will yield \$200 interest at 4%.

Start at the point 10 on the y-axis, or axis of time; go to the right to the \$200 interest graph; then drop to the axis of principals, the point thus reached being that of \$500. That is, the principal is \$500.

3. Find the approximate interest on \$900 for 9 yr. at 4%.

Follow the line leading from \$900 on the axis of principals to the point of intersection with the line leading from 9 on the axis of time. A line drawn through this point perpendicular to the adjacent \$300 and \$400 interest curves is divided at the point, approximately as 1 to 3. We say the interest is approximately \$325.

Exercise 152. Interest Graphs

Draw interest graphs, similar to those on page 260, for the following rates:

1. 3%. **2.** 5%. **3.** $5\frac{1}{2}$%. **4.** 6%.

5. From the graph on page 260 find what principal will yield \$500 in 12 yr. at 4%.

6. From the graph on page 260 find the time needed for \$2000 to earn \$400 interest at 4%.

7. From the graph on page 260 find what principal will earn as much interest in 10 yr. as \$100 will earn in $7\frac{1}{2}$ yr., the rate in each case being 4%.

8. From the graph on page 260 find what principal will earn \$300 interest in 2 yr. at 4%.

9. From your graph in Ex. 1 find the time needed for \$1500 to earn \$300 interest at 3%.

10. From your graph in Ex. 1 find what principal will earn \$300 interest in 10 yr. at 3%.

11. From your graph in Ex. 2 find the time needed for \$800 to earn \$200 interest at 5%.

12. From your graph in Ex. 2 find what principal will earn \$400 interest in 5 yr. at 5%.

13. From your graph in Ex. 3 find the time needed for \$500 to earn \$275 interest at $5\frac{1}{2}$%.

14. From your graph in Ex. 3 find what principal will earn \$200 interest in 3 yr. at $5\frac{1}{2}$%.

15. From your graph in Ex. 4 find the time needed for \$400 to earn \$100 interest at 6%.

16. From your graph in Ex. 4 find what principal will earn \$100 interest in 4 yr. at 6%.

17. Draw an interest graph similar to that in Ex. 1 for $4\frac{1}{2}\%$, and find the time needed for \$1000 to earn \$100 interest at this rate.

18. Draw an interest graph for $5\frac{3}{4}\%$ and solve the problem of Ex. 17 for this rate.

19. Using the formula $prt = i$, let $p = 1000$ and $i = 100$, so that we have $1000\,rt = 100$ or $rt = 0.1$. The graph of this equation will give a curve showing the relation of the rate to the time in which \$1000 will yield \$100 interest. Draw the graph of $rt = 0.1$, and a succession of graphs for the equations $1000\,rt = 200$, $1000\,rt = 300$, $\cdots$, $1000\,rt = 900$. State in what respects this family of graphs resembles, and in what respect it differs from, the one given on page 260.

20. Determine from the graphs constructed in Ex. 19 how much longer it will take \$1000 to earn \$300 at 3% than for \$1000 to earn \$200 at 4%.

21. Determine from the graphs of Ex. 19 how long it will take \$500 to earn \$400 interest at 3%.

22. Using the formula $a = p(1 + rt)$, construct the series of straight-line graphs for $r = 0.06$ and $t = \frac{1}{6}$.

23. Explain how the graph of Ex. 4 may be used to check the amount at maturity of a 60-day note bearing interest.

24. In the formula $a = p(1 + rt)$, if $a = 150$ and $p = 100$, what will be the general shape of the curve showing the relation of r and t?

25. Construct the graph of the formula mentioned in Ex. 24 and from it determine the time in which \$100 will amount to \$150 at 6%.

26. Construct on the diagram used in Ex. 25 the graph of the equations $i = 100 \times 0.05 \times t$ and $a = 100(1 + 0.05\,t)$.

Exercise 153. Review of Chapter XIII

1. Locate the points 2, 4, and -2, -4, and connect them by a straight line. What is the length of the connecting line? By what geometric principle can this length be determined without actual measurement?

2. What are the coordinates of the point of intersection of the graphs of the equations $y = \frac{1}{2}x + 3$ and $y = x$?

3. The cost equations of two different manufactured articles are $c = 4500 + 2.5\,n$ and $c = 3000 + 3\,n$. Find algebraically the value of n for which the cost of producing these two articles will be the same.

4. Solve Ex. 3 by drawing the graphs of the equations and determining the coordinates of the point of intersection.

5. If the cost of a wireless message of 20 words is 60¢, and that of one of 40 words is \$1, find by constructing the straight-line graph the cost of the first 10 words and the rate per word for the excess over 10 words.

6. Construct the graph of the formula $s = 16\,t^2$, which represents the space s passed over in t seconds by a body falling freely. From the graph determine the time it will take a body near the surface of the earth to fall 70 ft.

7. By drawing the graphs, determine whether the equations $x^2 + y^2 = 4$ and $y = x + 4$ are consistent.

8. Construct a series of curves whose equations are $xy = 4$, $xy = 5$, $xy = 6$, and so on. Explain how these curves can be used for multiplying and dividing numbers.

9. By drawing the graphs test the consistency of the equations $xy = 6$ and $x^2 + y^2 = 9$.

10. How can the graph of the second equation in Ex. 26, page 262, be easily constructed when the first one is drawn?

Exercise 154. Review of Chapters I–XIII

1. For what value of p will $i = prt$ become $i = rt$?

2. In the formula $i = prt$ what is the value of p in terms of t when $i = r$?

3. If $x = 3$, $y = 2$, $z = 5$, find the value of the expression $x^3 + 2xy - z^2 + 3\sqrt{5z}$.

4. Find the sum of $3a^2 - 4ab + 5b^2$, $6a^2 - 7ab - 9b^2$, $-11a^2 + 2ab - 8b^2$, and $4a^2 + 16ab - 9b^2$.

5. What must be added to $p + prt$ to make the sum $2p$?

6. Apply the principle $(a + b)(a - b) = a^2 - b^2$ to the multiplication of 2501 by 2499.

7. Using the principle $ab = ba$, find 87% of \$125.

8. Divide a by $100/p$.

9. Express the result of Ex. 8 as a rule for dividing any number by an aliquot part of 100.

10. In particular, in Ex. 9, if $p = 6$, what is the rule for dividing any number by $\frac{100}{6}$, or $16\frac{2}{3}$?

11. Derive a one-day rule for interest at $3\frac{3}{4}\%$. State the rule in three parts, as usual.

12. The catalogue price of pianos is \$1000 and the dealer receives discounts of 30, 10, 10. His cost of doing business is 23% of the selling price, and his rate of profit is 17% of the selling price. Find the selling price.

13. A city which owed \$19,000 was given the privilege of paying in three installments of any amounts, provided the total was paid in 3 yr. The city council desired to arrange the installments so that the second installment should be 50% more than the first, and the third 50% more than the second. Find the amounts, not considering interest.

INDEX

	PAGE
Abbreviations	1
Abscissa	245
Absolute value	58
Addition	61, 71
Aggregation, symbols of	16
Algebraic sum	72, 152
Aliquot parts	114
Antecedent	179
Area	10, 18, 25
Arranging a polynomial	100
Axes	245
Axioms	34
Balance sheet	29
Bar pictogram	43
Binomial	16
Board measure	12
Brokerage	202
Cartogram	50
Check	75, 78, 97, 102, 122
Circular pictogram	46
Clearing of fractions	163, 230
Coefficient	13
Cofactor	199
Commission	202
Common denominator	151
Compound interest	22
Consequent	179
Consignee	202
Coordinates	246
Cube	11, 13, 15, 20
root	14, 15
Daily balances	80
Decimals, equations involving	177
Denominator	149
Dietaries	237
Directed numbers	57
Discount	192, 206
Divisibility of numbers	126
Division	65, 119, 158
Elimination	222
Equation	29, 30, 131, 189, 221
Equivalent equations	250
Evaluating	3
Exact interest	216
Exponent	14, 95, 120
Extremes	183
Factor	13, 98, 108, 202
Formula	8, 10, 11, 12, 18, 22, 24, 113, 159, 167, 192, 203, 207
Fraction	149
Fractional equation	163, 230
Function	52
Graph	43, 52, 245, 248, 258
Gross cost	189
Identity	131
Inconsistent equations	250
Indeterminate equation	221
Integers, system of	68
Interest	22, 80, 206, 213, 258
Interpolation	254

	PAGE
Laws of exponents	95, 120
of proportion	183
of signs	88, 89, 94, 120, 151
Linear equation	131, 251
List price	189
Lowest common denominator	151
terms	150
Mean proportional	183
Means	183
Members of an equation	30
Mixed expression	154
Monomial	16
Multiple pictogram	45
Multiplication	65, 93, 94, 157
Negative numbers	57, 58
Net cost	189
Net-cost-rate factor	194
Net price	189, 192
Net proceeds	202
Net-profit graph	256
Nines, check of	102
Numerator	149
Numerical equation	132
One-day method	211
Order of operations	16
Ordinate	246
Origin	246
Parentheses	16, 88, 89
Percentage	38, 86
Pictogram	43
Plotting	246, 248
Polynomial	16
Positive numbers	58
Power	13, 20
Prime factor	13
Primes and subscripts	167
Principal	202, 206
Proceeds	207
Profit	189, 198
Proportion	183
Quadrants	246
Quantity, unknown	30
Ratio	179
Reciprocal	158, 199
Reduction of fractions	149, 154
Root	14, 132
Satisfying an equation	131
Short methods	124
Sign of a fraction	151
Signs, laws of	88, 89, 94, 120, 151
Similar monomials	71
Simple equation	131
Simplifying	3
Simultaneous equations	221
Six per cent method	209
Solid pictogram	46
Solvency, ratio of	241
Solving an equation	30, 131
Special products	105, 107, 111
Square	13, 15, 20, 105
root	14, 15
Subtraction	63, 71, 82
Symbols	1, 4, 15, 16, 58
Tables	20, 195, 200, 213, 215, 237, 252
Terms	16, 149, 179, 183, 192
Three unknowns	140, 234
Trade discount	192
Transposition	132
Trinomial	16
Two unknowns	134, 222
Unknown quantity	30
Zero	68

